# Macroscopic Models of Traffic Flow – A

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# Introduction

- Previously, two types of relationships among traffic flow characteristics were discussed:
  1. The flow-speed-density relationship or the identity:

$$q = k \times v$$

  - An identity is an equality that holds true regardless of the values chosen for its variables. They are often used in simplifying or rearranging algebra expressions: $(x + y)^2 = x^2 + y^2 + 2xy$.
  - It is location specific and time specific, that is, flow, speed, and density must refer to the same location and time : $q(t, x) = k(t, x) \times v(t, x)$

# Introduction

- Previously, two types of relationships among traffic flow characteristics were discussed:
  2. Pairwise relationships or equilibrium models:

$$v = V(k)$$

$$q = Q(k)$$

$$v = U(q)$$

These relations:

- Define the fundamental diagram
- They are location specific—that is, different locations and roads may have different underlying fundamental diagrams
- They are equilibrium models— that is, they describe a steady-state behavior in the long run, and hence are not specific to a particular time
- Such relationships are only of statistical significance—that is, the equal signs do not strictly hold in the real world.

$$\Rightarrow \quad \begin{aligned} v(x) &= V(k(x)) \\ q(x) &= Q(k(x)) \\ v(x) &= U(q(x)) \end{aligned}$$

# Traffic flow theory - Objective

- The main purpose of formulating a traffic flow theory is to help better understand traffic flow and, by the application of such knowledge, to control traffic for safer and more efficient operations.
- Hence, a good theory should be able to help answer the following questions:
  - Given existing traffic conditions on a road and upstream arrivals, how do road traffic conditions change over time?
  - Where are the bottlenecks, if any?
  - In the case of congestion, how long does it last and how far do queues spill back?
  - If an incident occurs, what is the best strategy so that the impact on traffic is minimized?
- Answers to these questions involve the analysis of dynamic change of traffic states over time and space.
- The equilibrium models are capable only of describing traffic states and do not provide a mechanism to analyze how such states evolve.
- This chapter introduces dynamic models to address these questions.

# The Continuity Equation

- The derivation of a dynamic equation starts with the examination of a small volume of roadway traffic as a continuum.

- Here traffic flow is treated as a one-dimensional compressible fluid like a gas.

- Conservation laws apply to this kind of fluid.

- The first-order form of conservation is mass conservation, also known as the continuity equation.

$$q_x + k_t = 0$$

- Here, we introduce 3 different perspectives to derive the continuity equations:
  - Derivation 1 – Finite Difference,
  - Derivation 2 – Fluid Dynamics,
  - Derivation 3: Three-Dimensional Representation of Traffic Flow.

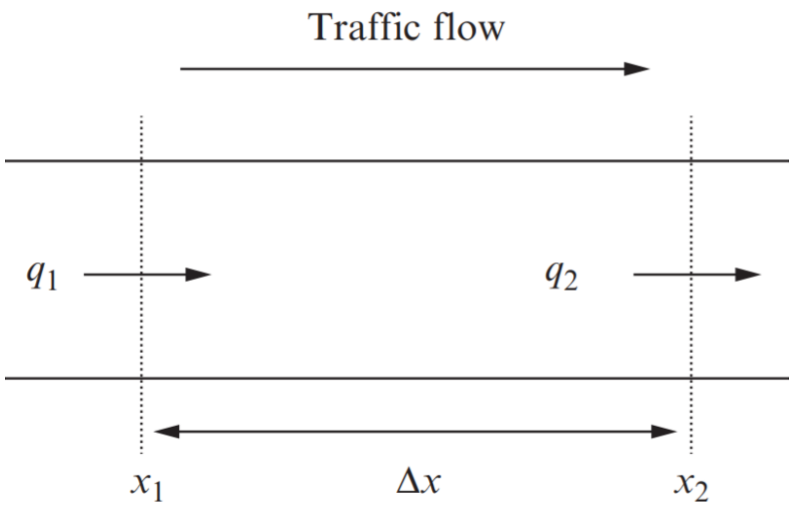
# Derivation 1 – Finite Difference

- Suppose a highway section is delineated by two observation stations at $x_1$ and $x_2$.
- Let $\Delta x = x_2 - x_1$ denote the section length.
- During time interval $\Delta t = t_2 - t_1$, $N_1$ vehicles passed $x_1$ and $N_2$ vehicles passed $x_2$. Therefore, the flow rates at these locations are:

$$q_1 = \frac{N_1}{\Delta t} \quad , \quad q_2 = \frac{N_2}{\Delta t}$$

- The change in the number of vehicles in the section is

$$\Delta N = N_2 - N_1 = (q_2 - q_1)\Delta t = \Delta q \Delta t$$




*Road section to derive the continuity equation*

# Derivation 1 – Finite Difference

- Assume the traffic densities in the section at $t_1$ and $t_2$ are $k_1$ and $k_2$, respectively.
- Therefore, there are $M_1 = k_1\Delta x$ vehicles in the section at time $t_1$ and $M_2 = k_2\Delta x$ vehicles in the section at time $t_2$ .
- The change in the number of vehicles in the section can be expressed as

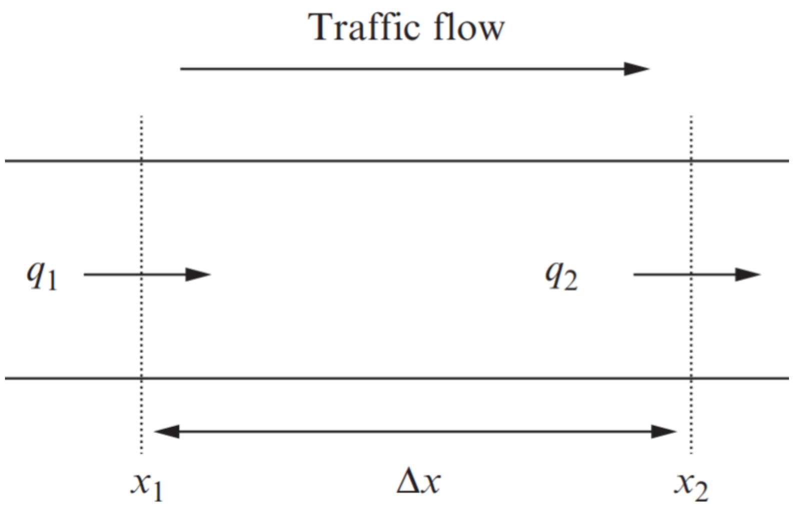
$$\Delta M = k_1\Delta x - k_2\Delta x = (k_1 - k_2)\Delta x = -\Delta k\Delta x.$$

- Since vehicles cannot be created or destroyed inside the section, the change in the number of vehicles should be the same in the same section during the same time interval.
  Therefore, $\Delta N = \Delta M$—that is:

$$\Delta q\Delta t = -\Delta k\Delta x, \quad \Rightarrow \quad \Delta q\Delta t + \Delta k\Delta x = 0$$

- Dividing both sides by $\Delta x\Delta t$, we get:

$$\frac{\Delta q}{\Delta x} + \frac{\Delta k}{\Delta t} = 0 \quad \xrightarrow[\Delta t \to 0]{\Delta x \to 0} \quad \frac{\partial q}{\partial x} + \frac{\partial k}{\partial t} = 0 \quad \Rightarrow \quad q_x + k_t = 0$$




*Road section to derive the continuity equation*

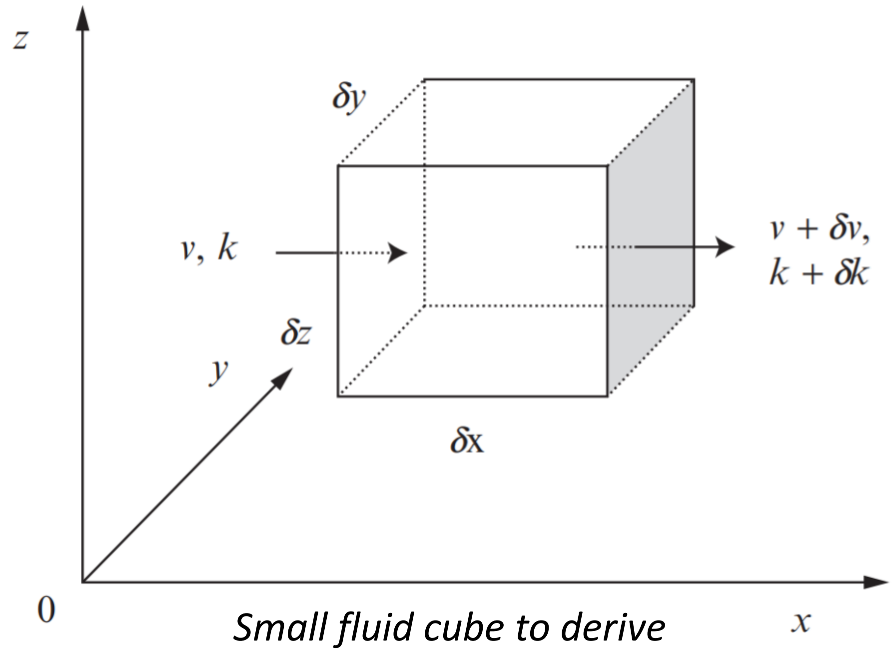
# Derivation 2: Fluid Dynamics

- Figure illustrates a small fluid cube of size $\delta x \times \delta y \times \delta z$
- The fluid velocity $v$ and density $k$ at two sides of the cube also are shown.
- The mass flow into the cube is $vk\delta y\delta z$.
- The mass flow out of the cube is:

$$(v+\delta v)(k+\delta k)\delta y\delta z = \left(v+\frac{\partial v}{\partial x}\delta x\right)\left(k+\frac{\partial k}{\partial x}\delta x\right)\delta y\delta z$$
$$= \left(vk + v\frac{\partial k}{\partial x}\delta x + k\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial x}\frac{\partial k}{\partial x}\delta x\delta x\right)\delta y\delta z$$

- The mass stored in the cube is equivalent to the mass that flows in minus mass that flows out:

$$\left(v\frac{\partial k}{\partial x}\delta x + k\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial x}\frac{\partial k}{\partial x}\delta x\delta x\right)\delta y\delta z$$
$$= \left(v\frac{\partial k}{\partial x} + k\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial k}{\partial x}\delta x\right)\delta x\delta y\delta z$$



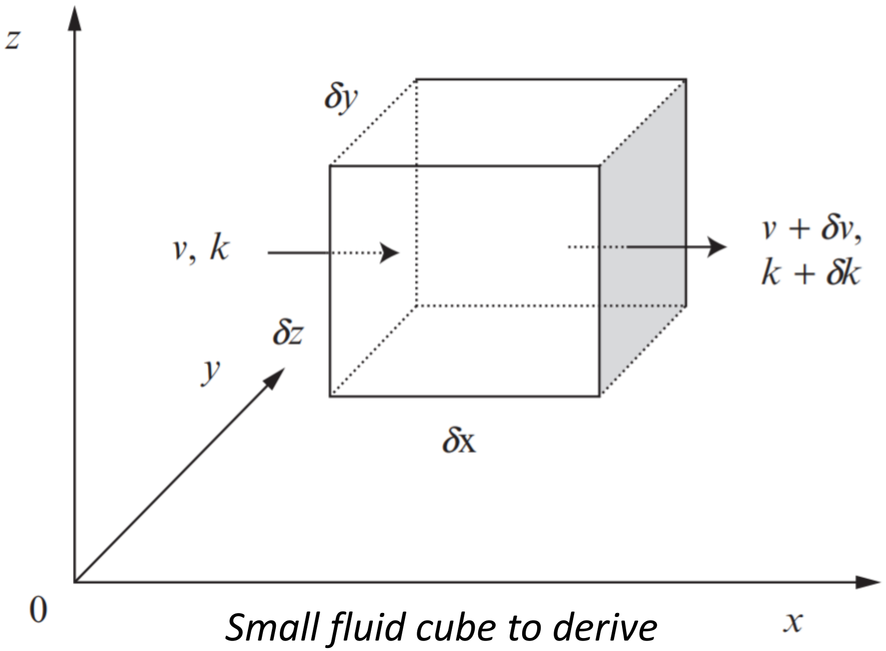

*Small fluid cube to derive the continuity equation*

# Derivation 2: Fluid Dynamics

- If we ignore the higher-order term, we have: $$\left(v\frac{\partial k}{\partial x}+k\frac{\partial v}{\partial x}\right)\delta x\delta y\delta z=\frac{\partial (kv)}{\partial x}\delta x\delta y\delta z$$
- Similar treatment applies to the other two directions of the cube, so the total mass stored in the cube is: $$\left(\frac{\partial (kv)}{\partial x}+\frac{\partial (ku)}{\partial y}+\frac{\partial (kw)}{\partial z}\right)\delta x\delta y\delta z$$
- The mass stored in the cube must be balanced by the change of mass in the cube: $$\frac{\partial k}{\partial t}\delta x\delta y\delta z$$
- The law of mass conservation requires that:

$$\left(\frac{\partial (kv)}{\partial x}+\frac{\partial (ku)}{\partial y}+\frac{\partial (kw)}{\partial z}\right)\delta x\delta y\delta z+\frac{\partial k}{\partial t}\delta x\delta y\delta z=0$$

$$\Rightarrow\quad \frac{\partial k}{\partial t}+\left(\frac{\partial (kv)}{\partial x}+\frac{\partial (ku)}{\partial y}+\frac{\partial (kw)}{\partial z}\right)=0$$




*Small fluid cube to derive the continuity equation*

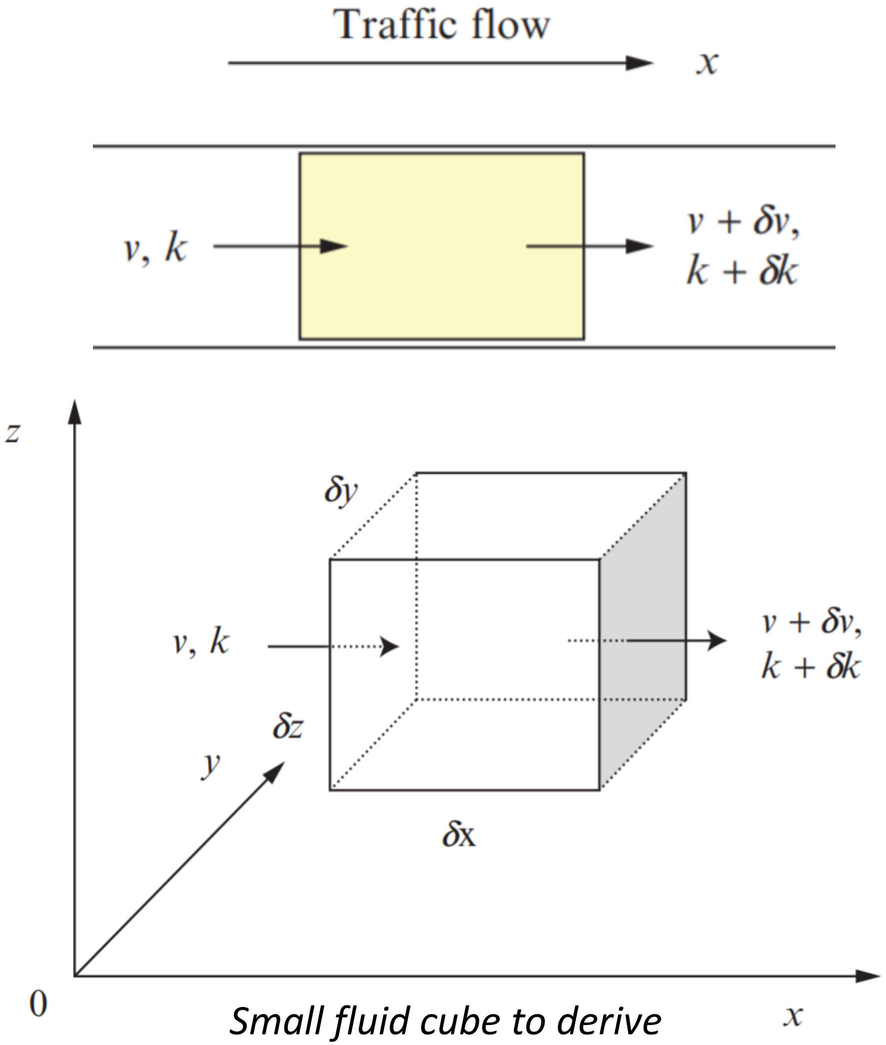
# Derivation 2: Fluid Dynamics

- Highway traffic constitutes a special case of the above situation with only one dimension:

$$\frac{\partial(kv)}{\partial x} + \frac{\partial k}{\partial t} = 0$$

- Note that $q = kv$. Therefore:

$$q_x + k_t = 0$$




*Small fluid cube to derive the continuity equation*

# Derivation 3: Three-Dimensional Representation

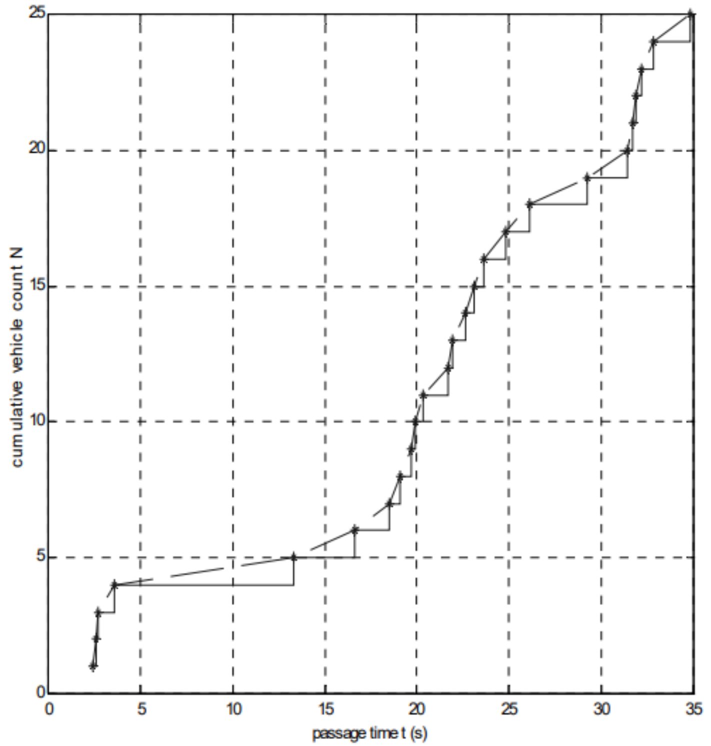
- As discussed previously, the surface which represents the cumulative number of vehicles, $N$, can be expressed as a function of time $t$ and space $x$—that is, $N = N(t, x)$.

- The density at time-space point $(t, x)$ is the first partial derivative of $N(t, x)$ with respect to $x$, but takes a negative value:

$$k(t, x) = -\frac{\partial N(t, x)}{\partial x}$$

- The flow at $(t, x)$ is the first partial derivative of $N(t, x)$ with respect to $t$:

$$q(t, x) = \frac{\partial N(t, x)}{\partial t}$$




*Cumulative flow function $N(x, t)$*

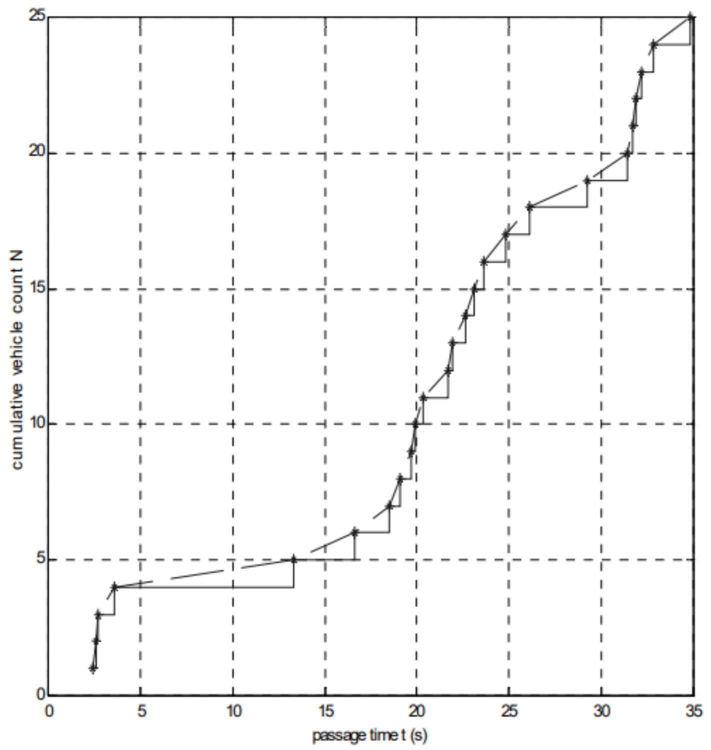
# Derivation 3: Three-Dimensional Representation

- If both the flow and the density have first-order derivatives:

$$\left.\begin{aligned}\frac{\partial q(t,x)}{\partial x} &= \frac{\partial N(t,x)/\partial t}{\partial x} = \frac{\partial N^2(t,x)}{\partial x \partial t} \\ \frac{\partial k(t,x)}{\partial t} &= \frac{-\partial N(t,x)/\partial x}{\partial t} = -\frac{\partial N^2(t,x)}{\partial x \partial t}\end{aligned}\right\} \Rightarrow \frac{\partial q(t,x)}{\partial x} = -\frac{\partial k(t,x)}{\partial t}$$

- Then:

$$q_x + k_t = 0$$




*Cumulative flow function $N(x,t)$*

# First-Order Dynamic Model

- Traffic evolution is the process of how traffic states (e.g., flow $q$, speed $v$, and density $k$) evolve over time $t$ and space x given some initial conditions (e.g., $k_0 = k(0, x)$)and boundary conditions (e.g., $q(t) = q(t, x_0)$).
- One recognizes that time $t$ and space $x$ are independent variables and traffic states are dependent variables—that is, they are functions of time and space ($q = q(t, x), v = v(t, x), k = k(t, x)$).
- The continuity equations derived before dynamically relate the change of flow $q_x$ to the change of density $k_t$

$$q_x + k_t = 0$$

- This equation contains two unknown variables $q(t, x)$ and $k(t, x)$. Since the number of unknown variables is greater than the number of equations, the problem is underspecified. Because of this, another simultaneous equation is needed.
- Hopefully, the identity comes handy:

$$q(t, x) = k(t, x) \times v(t, x)$$

# First-Order Dynamic Model

- By adding this new equation, we introduce a third unknown variable— that is, speed $v(t, x)$.
- Therefore, a third simultaneous equation is called for.
- We are unable to find a third governing equation that holds for any time and space.
- Consequently, we have to accept the (less-than-ideal) option by looking at equilibrium traffic flow models (e.g., the Greenshields model), which are known to hold only statistically. Such a model takes the form of: $v = V(k)$
- Putting everything together, one obtains a system of three equations involving three unknown variables:

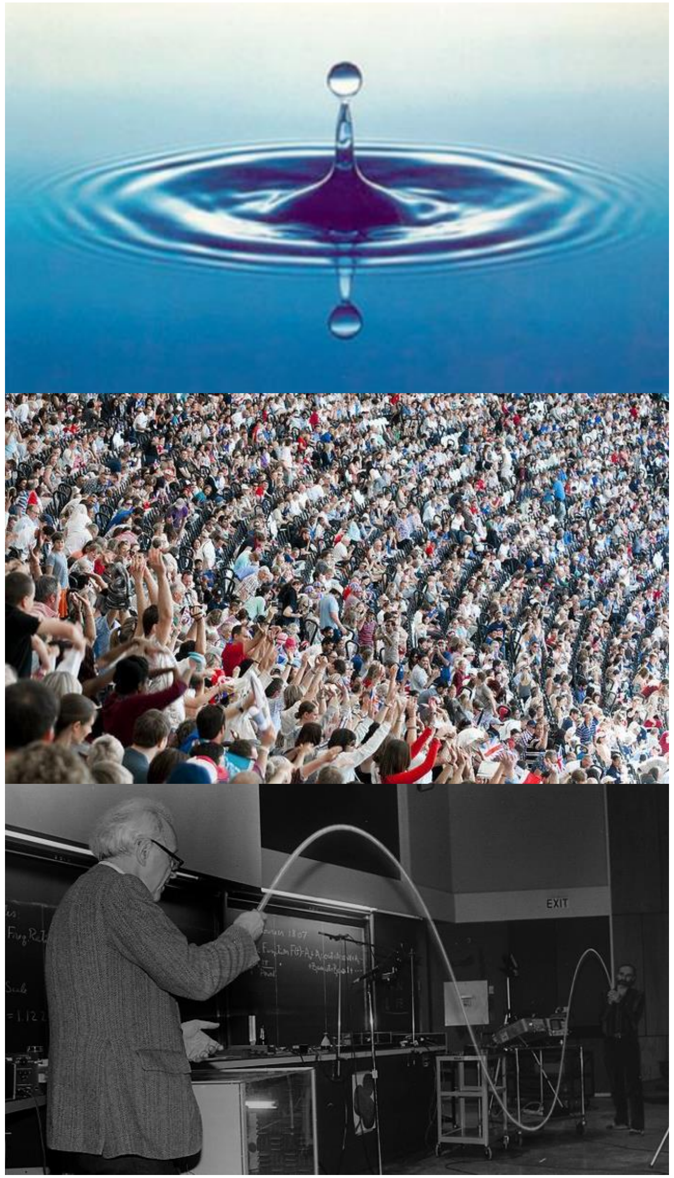
$$\begin{cases} q_x + k_t = 0, \\ q = kv, \\ v = V(k). \end{cases}$$

- If initial and boundary conditions are provided, the above system of equations may be solvable.
- One can determine the traffic state at an arbitrary time-space point $(t, x)$—that is, $q(t, x)$, $v(t, x)$, and $k(t, x)$. and answer the questions posed at the beginning of this section.
- However, solving such a system of equations is not easy!

# Wave Phenomena

- Waves are everywhere in the real world.
    - In a pond
    - A football stadium
    - Shaking a rope
- A wave is the propagation of a disturbance in a medium over time and space.
- If we apply the notion to a platoon of vehicles on a highway, when one of the vehicles brakes suddenly and then resumes its original speed, subsequent vehicles will be affected successively.
- The propagation of such a disturbance is a wave, and the traffic is the medium.

*Traffic waves on a highway*




*Waves*

# Mathematical Representation

- Waves are described by partial differential equations (PDEs). For example: $k_{tt} = ak_{xx}$

- Note that:
  - If a dependent variable $k$ is a function of independent variables $t$ and $x$, we write $k = k(t,x)$ and we denote its partial derivatives with respect to $x$ and $t$ as follows:

$$k_x = \frac{\partial k}{\partial x}, k_t = \frac{\partial k}{\partial t}, k_{xt} = \frac{\partial^2 k}{\partial x \partial t}, k_{tx} = \frac{\partial^2 k}{\partial t \partial x}, k_{xx} = \frac{\partial^2 k}{\partial x^2}, k_{tt} = \frac{\partial^2 k}{\partial t^2}$$

  - A PDE for $k(t,x)$ is an equation that involves one or more partial derivatives of $k$ with respect to $t$ and $x$, for example:

$$k_t = k_x + k, k_t = k_{xx} + k_x + 5, k_t = k_{xxx} + 4k + \cos x.$$

# Mathematical Representation

➢ PDEs can be classified on the basis of their order, homogeneity, and linearity:

## Order

➢ The order of a PDE is the order of the highest partial derivative in the equation.
For example:

- *first-order* PDE: $k_t = k_x + k$;
- *second-order* PDE: $k_t = k_{xx} + k_x + 5$;
- *third-order* PDE: $k_t = k_{xxx} + 4k + \cos x$.

➢ A first-order PDE can be expressed in the following general form:

$$P(t, x, k)k_t + Q(t, x, k)k_x = R(t, x, k)$$

- Where $P$, $Q$, and $R$ are coefficients, and they may be functions of $t$, $x$, and $k$.

## Homogeneity

➢ A first-order PDE $P(t,x,k)k_t + Q(t,x,k)k_x = R(t,x,k)$ is

- homogeneous if $R(t,x,k) = 0$;
- nonhomogeneous if $R(t,x,k) = 0$.

# Mathematical Representation

Linearity

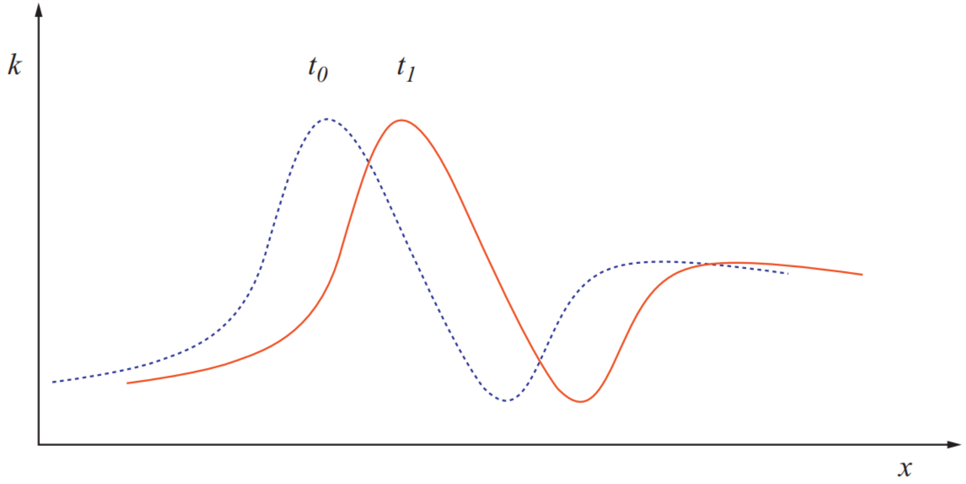
- In the above general first-order PDE, if both $P$ and $Q$ are independent of $k$, that is, $P = P(t,x), Q = Q(t,x)$, and
  - If $R$ is also independent of $k$—that is, $R = R(t,x)$—then the PDE is strictly linear. For example, $2xk_t + 3k_x = 5t$.
  - If $R$ is linearly dependent on $k$—that is, $R = R(t,x,k)$—then the PDE is linear. For example, $2xk_t + 3k_x = 5k + 3$.
  - If $R$ is dependent on $k$ in a nonlinear manner, then the PDE is semi-linear. For example, $2xk_t + 3k_x = e^k$.

- If $P$ or $Q$ is dependent on $k$, or both $P$ and $Q$ are dependent on $k$, that is, $P = P(t,x,k), Q = Q(t,x,k)$, and $R = R(t,x,k)$, then the PDE is quasilinear. For example, $k_t + (3k + 2)k_x = 0$.

- A PDE is nonlinear if it involves cross terms of $k$ and its derivatives, for example, $k_t k_x + k = 2$.

# Traveling Waves

- Many PDEs have solutions in a traveling wave form : $k(t,x) = f(x - ct)$
- Figure illustrates two instants of the traveling wave, $f\ (x - ct_0)$ and $f\ (x - ct_1)$.
- Note that:
  1. The traveling wave preserves its shape
  2. The wave at time $t_1$ is simply a horizontal translation of its initial profile at time $t_0$
- If $c$ is a positive constant, wave $k(t,x) = f\ (x - ct)$ travels to the right over time, while wave $k(t,x) = f\ (x + ct)$ moves to the left.




*A traveling wave*

# Traveling Wave Solutions

- To solve the following wave equation where $a$ is a constant : $k_{tt} = ak_{xx}$
- Assume that a solution to the above wave equation takes a traveling form $k(t,x) = f\,(x - ct)$. Let $z = x - ct$. Then:

$$k_t = \frac{\partial k}{\partial t} = \frac{\mathrm{d}f}{\mathrm{d}z}\frac{\partial z}{\partial t} = f' \times (-c) = -cf'$$

- Similarly: $k_x = f'$, $k_{tt} = c^2 f''$, and $k_{xx} = f''$

- Plugging the above expressions into the wave equation, one obtains: $(c^2 - a)f'' = 0$

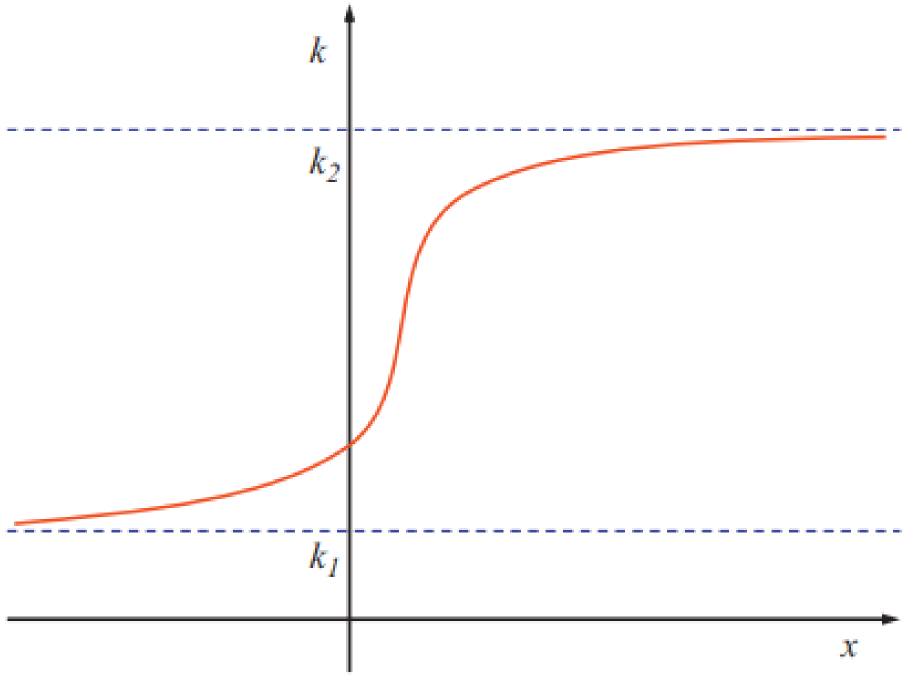
- There are two ways for the left-hand side to be $0$:
  1. If $c^2 - a = 0$, then $k(t,x) = f\,(x \pm \sqrt{a}t)$, where $f$ can take any functional form.
  2. If $f = 0$, then $k(t,x) = A + B(x - ct)$, where $A$ and $B$ are arbitrary constants.

# Wave Front And Pulse

- A traveling wave is called a wave front if:

$$\begin{cases} k(t,x) = k_1 & \text{as } x \to -\infty \\ k(t,x) = k_2 & \text{as } x \to +\infty \end{cases}$$

- A traveling wave is called a pulse if $k_1 = k_2$




*A wave front*

# General Solution to Wave Equations

- As previously mentioned, the wave equation in one dimension is expressed by:

$$k_{tt} = ak_{xx}$$

- Many wave equations have a general solution in the form of superposition of traveling waves:

$$k(t, x) = F(x - ct) + G(x + ct)$$

- Note that even though each of the terms on the right-hand side is a traveling wave, their superposition may not necessarily be.

# General Solution to Wave Equations

Method of change of variables:

$$u_{tt} = c^2 u_{xx} \rightarrow \frac{\partial^2 u}{\partial t^2} = a\frac{\partial^2 u}{\partial x^2}$$

→ *Define two new coordinates* $r = x + ct, s = x - c\,t$

*Converting our partial derivatives in t and x to partial derivatives in r and s using the chain rule* →

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial x} \\[2ex] \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial t} + \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial t} \end{cases}$$

$$\rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial s \partial r}\frac{\partial s}{\partial x} + \frac{\partial^2 u}{\partial r \partial s}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial s^2}\frac{\partial s}{\partial x} = \frac{\partial^2 u}{\partial r^2} + 2\frac{\partial^2 u}{\partial s \partial r} + \frac{\partial^2 u}{\partial s^2}$$

$$\frac{\partial u}{\partial t} = c\frac{\partial u}{\partial r} - c\frac{\partial u}{\partial s} \rightarrow \frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial t} + c\frac{\partial^2 u}{\partial s \partial r}\frac{\partial s}{\partial t} - c\frac{\partial^2 u}{\partial r \partial s}\frac{\partial r}{\partial t} - c\frac{\partial^2 u}{\partial s^2}\frac{\partial s}{\partial t}$$

$$= c^2\frac{\partial^2 u}{\partial r^2} - 2c^2\frac{\partial^2 u}{\partial s \partial r} + c^2\frac{\partial^2 u}{\partial s^2}$$

*Apply the chaine rule again to find the second derivatives*

*Plug into the wave equation:* →

$$c^2\frac{\partial^2 u}{\partial r^2} - 2c^2\frac{\partial^2 u}{\partial s \partial r} + c^2\frac{\partial^2 u}{\partial s^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + 2\frac{\partial^2 u}{\partial s \partial r} + \frac{\partial^2 u}{\partial s^2}\right) \rightarrow 4c^2\frac{\partial^2 u}{\partial s \partial r} = 0 \rightarrow \frac{\partial^2 u}{\partial s \partial r} = 0$$

# General Solution to Wave Equations

## Method of change of variables:

$$\frac{\partial^2 u}{\partial s \partial r} = 0$$

*Integrate with respect to r* ➔

$$\frac{\partial u}{\partial s} = h(s)$$

*h is a pure function of s*

➔ *Integrate with respect to s* ➔

$$u = f(r) + g(s)$$

*f is a pure function of r*

*The anti-derivative of h(s)*

*We also had:* $r = x + ct, s = x - c\,t$

*The general solution to the wave equation*

$$u(x,t) = f(x+ct) + g(x - c\,t)$$

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*Suppose we choose a fixed point f(0) on the function f.* $x + ct = 0$ ➔ $x = -ct$ (*time*: $t$; *Wave speed*: $c$)

*So as time increases, the horizontal position of f(0) goes backwards* ➔ *Since f(0) has a fixed value, a point of fixed value is moving backwards in space* ➔ *In fact, all the othe points such as f(1), f(-1),etc. are also moving backwards at the same speed* ➔ $f(x+ct)$ *Backward travelling wave*

*With the same logic* ➔ $g(x - c\,t)$ *Forward travelling wave* ➔

*The general solution to the wave equation is a backward travelling wave supperposed by a forward travelling wave*

# D'Alembert Solution to the Wave Equation

Solution in a specific infinite one-dimensional domain

$$u_{tt} = c^2 u_{xx} \Rightarrow \frac{\partial^2 u}{\partial t^2} = a\,\frac{\partial^2 u}{\partial x^2}$$

$$\begin{cases} -\infty < x < +\infty, t > 0 & \text{No explicit boundary conditions} \\ u(x,0) = u_0(x) & \text{Some initial displacement} \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = v_0(x) & \text{Some initial velocity} \end{cases}$$

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Solution

*The general solution to the wave equation*

*f and g are unknowns*

$$u(x,t) = f(x+ct) + g(x-c\,t)$$

*Substitute the initial conditions*

$$\overset{(1)}{\Rightarrow} \quad u(x,0) = f(x+c0) + g(x-c\,0) = f(x) + g(x) \qquad (A)$$

$$\overset{(2)}{\Rightarrow} \frac{\partial u}{\partial t} = f'(x+ct)c + g'(x-c\,t)(-c) \Rightarrow \left.\frac{\partial u}{\partial t}\right|_{t=0} = v_0(x) = cf'(x) - cg'(x)$$

*Note that:* $\dfrac{\partial f(g(x))}{\partial x} = f'(g(x))g'(x)$

*Integrate from $x_0$ to $x$* (*Dummy integration variable*: $s$)

$$\int_{x_0}^{x} v_0(s)\,ds = c[f(x) - f(x_0)] - c[g(x) - g(x_0)]$$

$$\Rightarrow \frac{1}{c}\int_{x_0}^{x} v_0(s)\,ds + f(x_0) + g(x_0) = f(x) - g(x) \qquad (B)$$

# D'Alembert Solution to the Wave Equation

Solution in a specific infinite one-dimensional domain

$(A) + (B)$

➔

*And replace x by x+ct to get an expression for the backwards travelling wave*

$$f(x+ct) = \frac{1}{2}u_0(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} v_0(s)\,ds + \frac{f(x_0)-g(x_0)}{2}$$

$(A) - (B)$

➔

*And replace x by x-ct to get an expression for the forward travelling wave*

$$g(x-ct) = \frac{1}{2}u_0(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} v_0(s)\,ds + \frac{g(x_0)-f(x_0)}{2}$$

*The general solution to the wave equation*

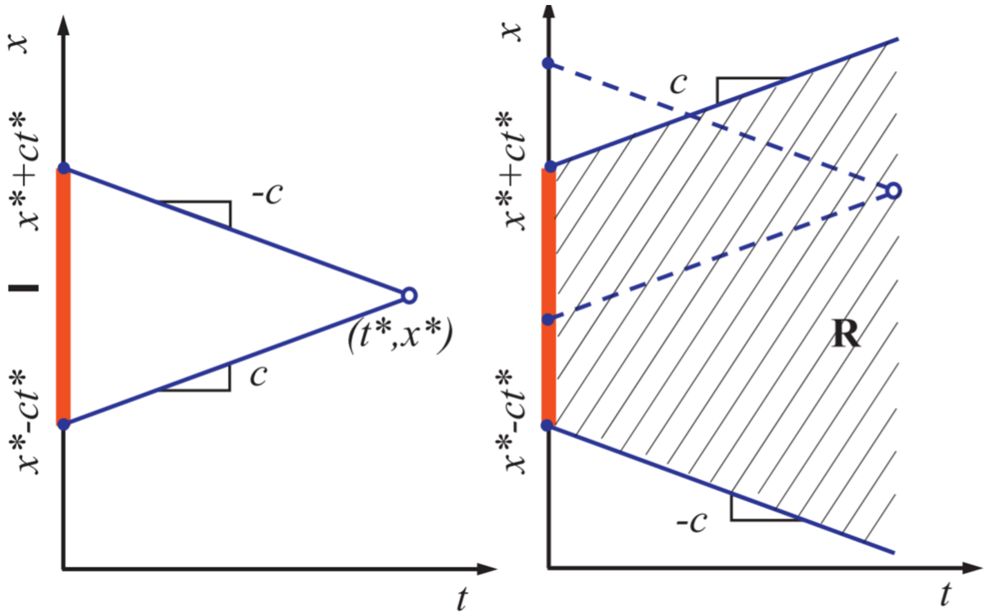
$$u(x,t) = f(x+ct) + g(x-c\,t) = \frac{1}{2}[u_0(x+ct) + u_0(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(s)\,ds$$

$$u(x,t) = \frac{1}{2}[u_0(x+ct) + u_0(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(s)\,ds$$

*D'Alembert Solution to the Wave Equation*

# Characteristics

- Therefore, it has been shown that if: $f(x) = k(x,0)$ and $g(x) = k_t(x,0)$
  The solution takes the following form:
  $$k(t,x) = \frac{1}{2}[k(x-ct,0) + k(x+ct,0)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\mathrm{d}s$$
- Applying the above conclusion, one notices that the solution $k$ at an arbitrary time-space point $(t^*, x^*)$ is:
  $$k(t^*,x^*) = \frac{1}{2}[k(x^*-ct^*,0) + k(x^*+ct^*,0)] + \frac{1}{2c}\int_{x^*-ct^*}^{x^*+ct^*} g(s)\mathrm{d}s$$
- The above equation suggests that the solution at an arbitrary point $(t^*, x^*)$ can be determined by
  - The initial condition at points $(0, x^* - ct^*)$ and $(0, x^* + ct^*)$
  - And, the interval $I$ bounded by the two points (inclusive)—that is, $I = [x^* - ct^*, x^* + ct^*]$.
- This is illustrated in the left part of the Figure . Therefore, the interval $I$ is called the *domain of dependence* of point $(t^*, x^*)$.



*Characteristics*

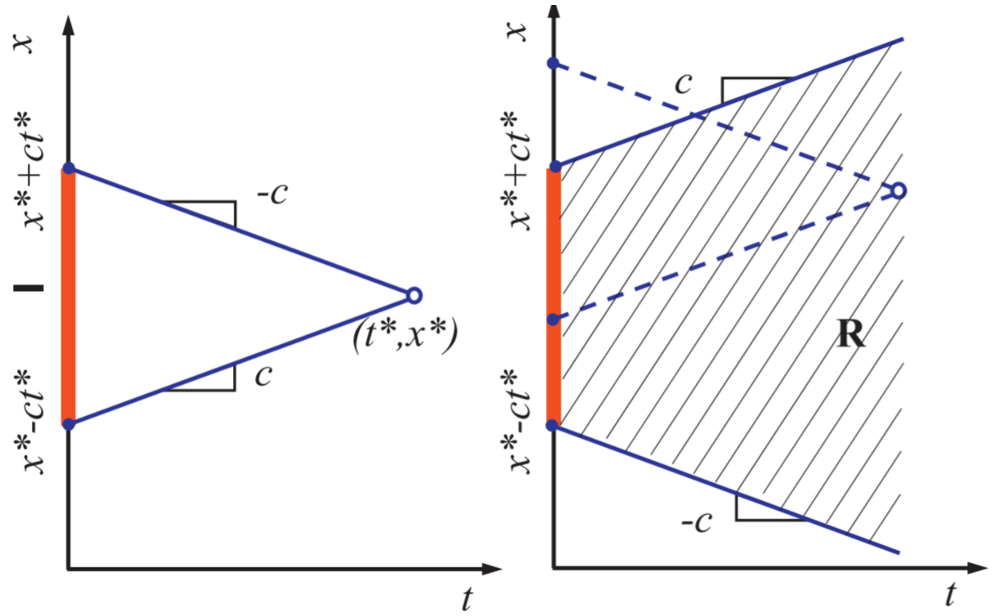
# Characteristics

- Notice that in the left part of the figure, the two lines coming from point $(t^*, x^*)$ intersecting the $x$-axis at $(0, x^* - ct^*)$ and $(0, x^* + ct^*)$ have slopes $c$ and $-c$. These two lines are called ***characteristic lines*** or simply ***characteristics.***
- The term "range of influence" refers to a collection of time-space points whose solutions are influenced either completely or partially by the domain of dependence $I$; see the shaded area in the right part of the figure.

- When $k_t(0, x) = 0$ the solution of the wave equation reduces to:

$$k(t, x) = \frac{1}{2}[k(0, x - ct) + k(0, x + ct)]$$

- This shows that the value of $k$ at $(t, x)$ depends only on the initial values of $k$ at two points, $x_1 = x - ct$ and $x_2 = x + ct$.
- Once the initial values $k(0, x - ct)$ and $k(0, x + ct)$ are known, one constructs the solution $k$ at $(t, x)$ by taking the average of $k(0, x_1)$ and $k(0, x_2)$.




*Characteristics*

# Characteristics

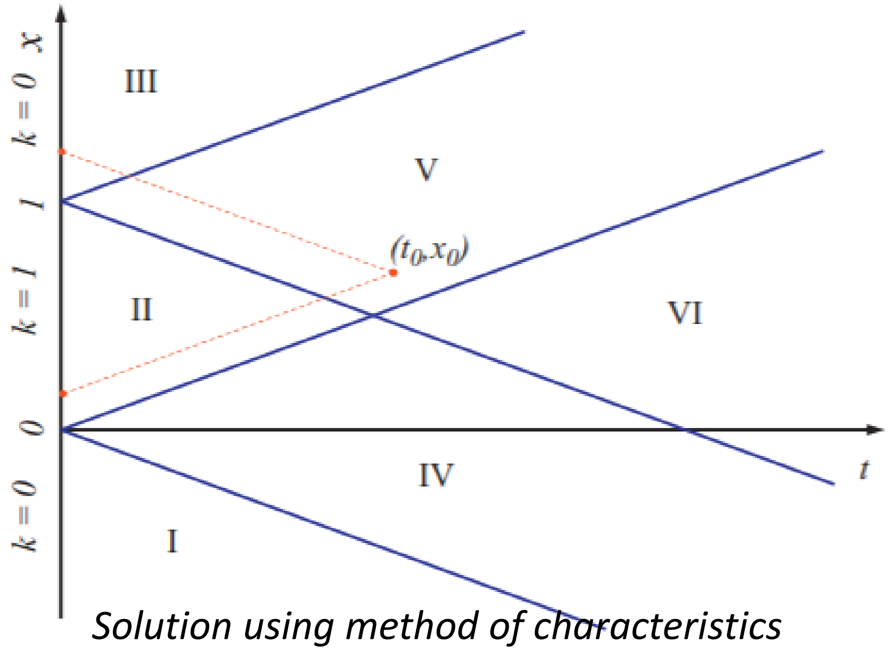
## Example

- Use characteristics to solve the following wave equation:

$$\begin{cases} k_{tt} = 4k_{xx}, \\ k(0,x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \text{ or} \\ 0 & \text{otherwise,} \end{cases} \\ k_t(0,x) = 0, \\ -\infty < x < +\infty, t > 0. \end{cases}$$

## Solution

- In this equation, the traveling wave speed $c = \pm 2$—that is, $k(t,x) = f(x \pm 2t)$.
- First, construct an $x - t$ plane.
- Locate points 0 and 1 on the x-axis.
- Then draws two characteristics (their slopes are $\pm 2$) from each of the two points.
- The four characteristics partition the $x - t$ plane into six regions as labeled in Figure.




*Solution using method of characteristics*

# Characteristics

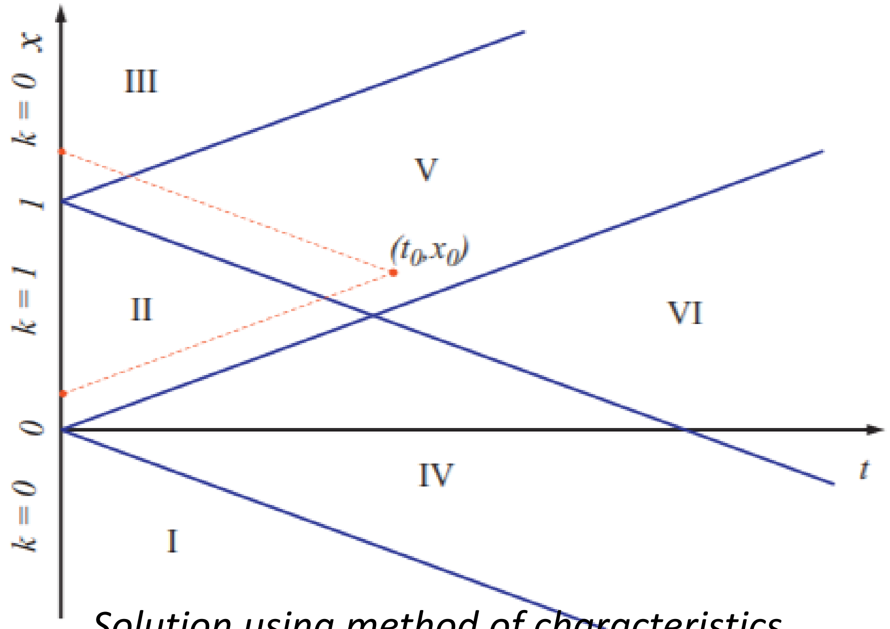
## Example

- Use characteristics to solve the following wave equation:

$$\begin{cases} k_{tt} = 4k_{xx}, \\ k(0,x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \text{ or} \\ 0 & \text{otherwise,} \end{cases} \\ k_t(0,x) = 0, \\ -\infty < x < +\infty, t > 0. \end{cases}$$

## Solution

- Take an arbitrary point $(t_0, x_0)$.
- The solution at this point is found by drawing two characteristics from this point.
- Then find the intersections of the two characteristics on the $x$-axis.
- Next, find the $k$ values at the two intersections. In this case the $k$ values are 1 and 0.
- Then the solution $k$ at point $(t_0, x_0)$ is the average of the $k$ values at the two intersections, that is, $k(t_0, x_0) = 1/2$.




*Solution using method of characteristics*

# Characteristics

## Example

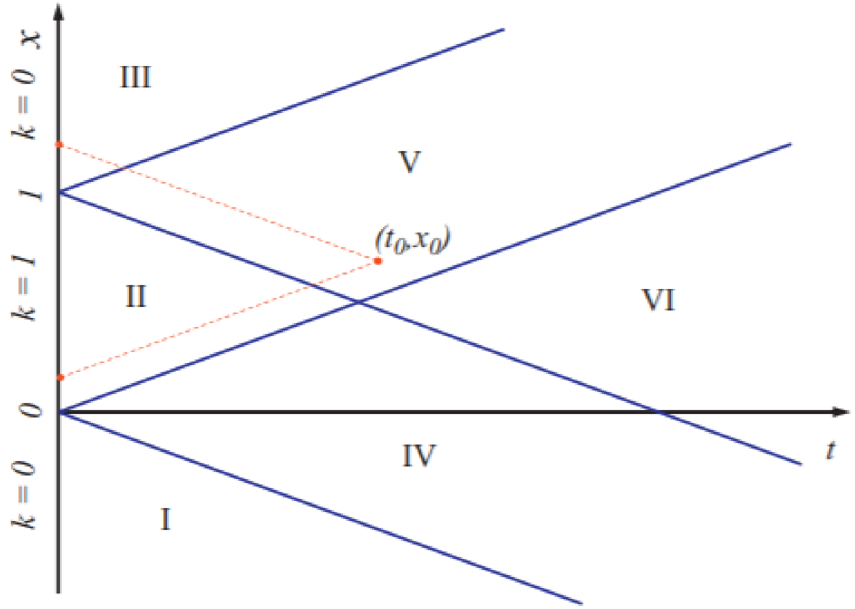
- Use characteristics to solve the following wave equation:

$$\begin{cases} k_{tt} = 4k_{xx}, \\ k(0,x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \text{ or} \\ 0 & \text{otherwise,} \end{cases} \\ k_t(0,x) = 0, \\ -\infty < x < +\infty, t > 0. \end{cases}$$

## Solution

- With use of a similar technique, the solution in other regions can be determined.

$$k(t,x) = \begin{cases} 0 & \text{if } (t,x) \in \text{region I}, \\ 1 & \text{if } (t,x) \in \text{region II}, \\ 0 & \text{if } (t,x) \in \text{region III}, \\ \frac{1}{2} & \text{if } (t,x) \in \text{region IV}, \\ \frac{1}{2} & \text{if } (t,x) \in \text{region V}, \\ 0 & \text{if } (t,x) \in \text{region VI}. \end{cases}$$




*Solution using characteristics*

# Method of Characteristics

- Let's consider a very simple PDE derived from the conservation law with an initial condition: $k_t + q_x = 0$

- If one assumes $q = ck$, where c is a constant, then $q_x = ck_x$, and the PDE can be defined as follows:

$$\begin{cases} k_t + ck_x = 0, \\ k(0, x) = k_0(x), \\ -\infty < x < \infty, 0 < t, \\ c \text{ is a constant.} \end{cases} \qquad (A)$$

- The goal is to find a solution to this PDE ➔ This is equivalent of finding the value of $k$ at an arbitrary time-space point, $k(t, x)$.
- Rather than working on an arbitrary point in the entire time-space plane, one starts with a simpler case by working on a point on a specific curve in the time-space plane.
- To do this, let's assume a curve $x = x(t)$.

# Method of Characteristics

- Therefore, the new goal is to find the value of $k$ at an arbitrary point $(t, x(t))$ on the curve—that is, $k\big(t, x(t)\big)$, and examine how $k$ changes along the curve $x = x(t)$.

- The rate of change of $k$ with time is the first derivative of $k$ with respect to time $t$; that is:

$$\frac{\mathrm{d}k(t, x(t))}{\mathrm{d}t} = \frac{\partial k}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}t} + \frac{\partial k}{\partial x}\frac{\mathrm{d}x(t)}{\mathrm{d}t} = k_t + \frac{\mathrm{d}x}{\mathrm{d}t}k_x \qquad (B)$$

- Comparing $(A)$ and $(B)$ we have:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = c \quad \Rightarrow \quad \frac{\mathrm{d}k(t, x(t))}{\mathrm{d}t} = k_t + ck_x = 0$$

- This means that the total time derivative of $k$ along the curve $x = x(t)$ is zero ➔ The value of $k$ is constant along the curve.

- This implies that the curve $x = x(t)$ needs to be drawn such that it is a straight line with slope of $c$.

# Method of Characteristics

- To find the equation of the line, we need to solve the following ordinary differential equation:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = c \quad \xrightarrow{\text{Integrate with respect to } t} \quad x(t) = ct + x_0$$

- At time $t = 0$, this line intersects the $x$-axis at $x_0$.
- Since $k$ remains constant along this line, the solution $k$ at any point on this line, $k(t, x(t))$, is the same as $k(0, x_0) = k(x_0)$, which is given in the initial condition.
- Therefore, we have found the solution for all points on this line. Such a line is called a *characteristic*.
  It is the same as the characteristic previously explained, where it is a line drawn from a time-space point with slope c, which is the speed of the traveling wave $f(x - ct)$.
- Since a characteristic denotes a set of time-space points on which the solution of $k$ remains constant, $k$ may be multivalued at the intersection of two characteristics. Such an occurrence is called a *gradient catastrophe*.

# Method of Characteristics

Recap

- the method of characteristics was discussed as a means to solve the continuity equation (i.e., conservation law) with an initial condition:

$$\begin{cases} k_t + q_x = 0, \\ k(0, x) = k_0(x), \\ -\infty < x < \infty, 0 < t, \end{cases}$$

- Where $q = Q(k)$ is a function of $k$:

$$q_x = \frac{\partial q}{\partial x} = \frac{\partial Q(k)}{\partial x} = \frac{dQ}{dk}\frac{\partial k}{\partial x} = Q'(k)k_x = ck_x$$

- To find the solution of $k$ at an arbitrary time-space point $(t^*, x^*)$, $k(t^*, x^*)$, one simply constructs a characteristic $x = ct + x_0$ which starts from $(t^*, x^*)$ and extends back to the x-axis at intercept $(0, x^* - ct^*)$.

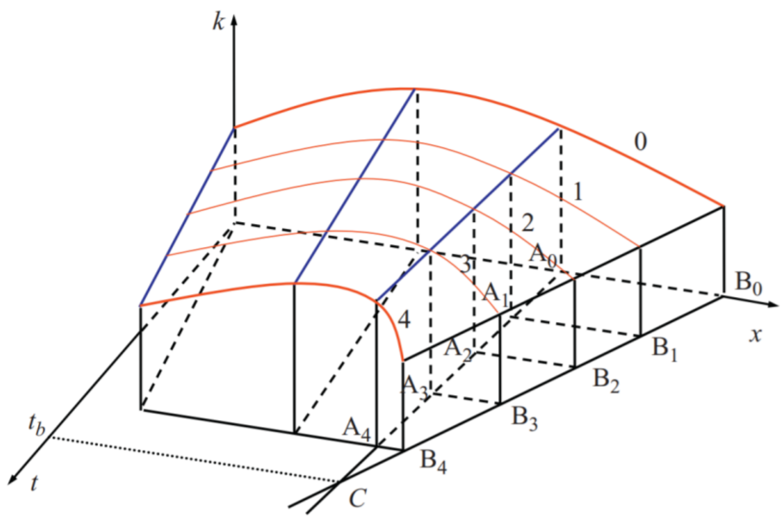
- Since $k((0, x^* - ct^*)) = k_0(x^* - ct^*)$ is given in the initial condition and $k$ remains constant along the characteristic, the solution is:

$$k(t^*, x^*) = k_0(x^* - ct^*)$$

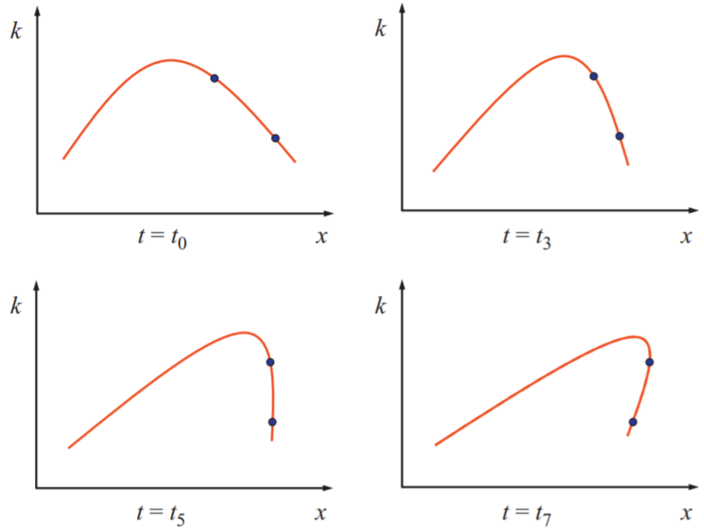
# Gradient Catastrophes

- In this solution, if $c$ is constant, characteristics drawn from two different time-space points remain straight, parallel lines.
  - Hence, any time-space point lies on one and only one characteristic, and the solution at this point is single valued.

- However, if $c$ is a function of $k$, $c = c(k)$, and not explicitly dependent on $x$ or $t$, two different characteristics drawn from two time-space points are still straight lines but they may not necessarily be parallel, in which case they may intersect and the solution at this intersection may be multivalued.



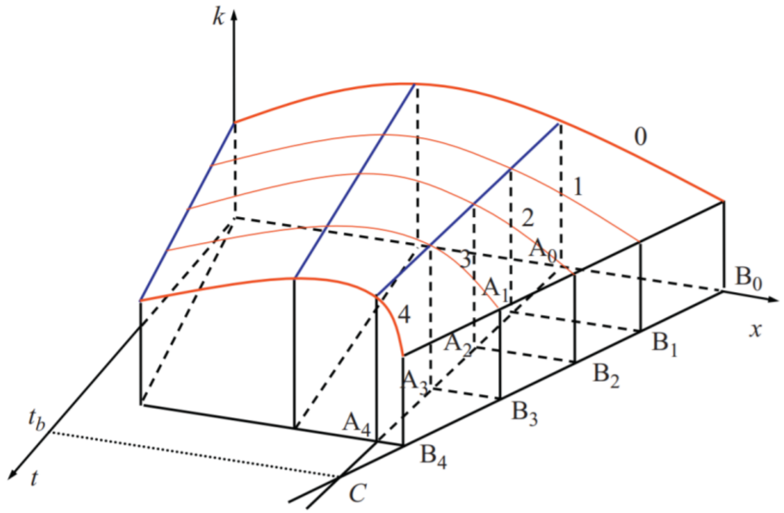

*A gradient catastrophe*



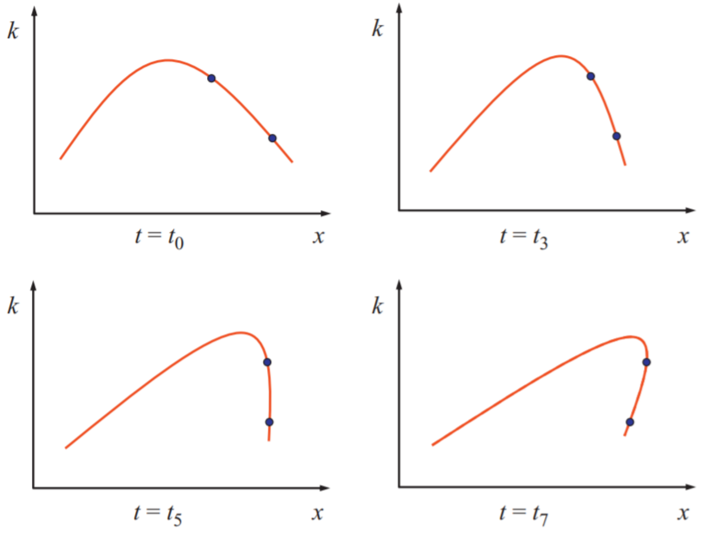

*Solution profile illustrating gradient catastrophe*

# Gradient Catastrophes

- As the two characteristics become closer and closer, the gradient of the solution profile (red curves above the characteristics) becomes increasingly steep.
- At the intersection of two characteristics (point $C$), the solution profile will have an infinite gradient at this point.
- The formation of such an infinite gradient is called a *gradient catastrophe.*
- The time when infinite gradient occurs is called the *break time* $t_b$.
- After this point, the solution profile ceases to be a valid function, and the solution beyond the break time will be problematic.
- In this section, we are trying to address such an issue.




*A gradient catastrophe*

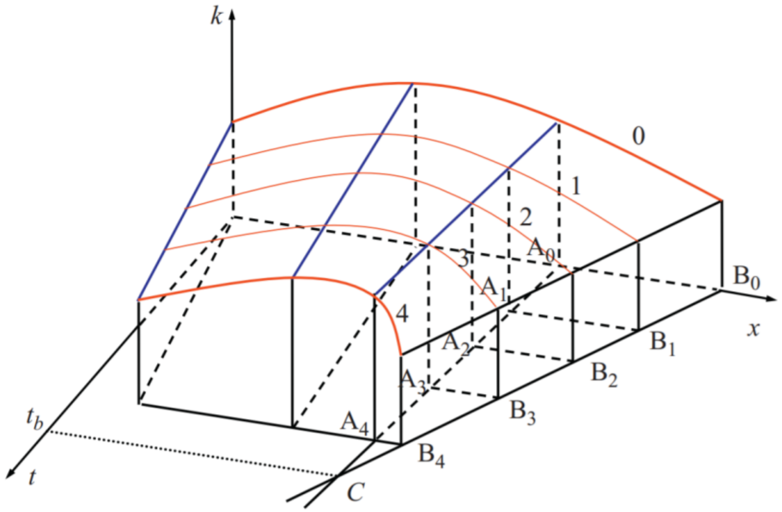



*Solution profile illustrating gradient catastrophe*

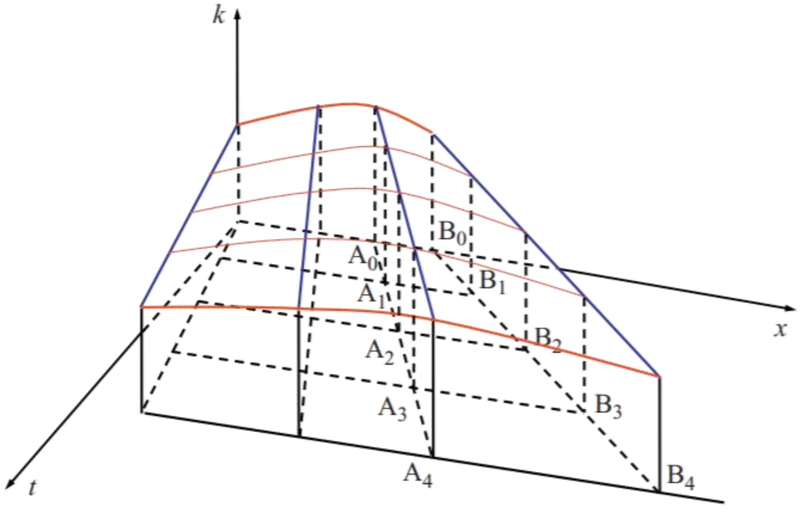
# Gradient Catastrophes

- The example below (left) illustrates a family of characteristics moving closer and closer over time, so they form a ***compression wave***.

- The opposite case is a family of characteristics moving farther and farther apart without any intersection; such a wave is called an ***expansion wave***.

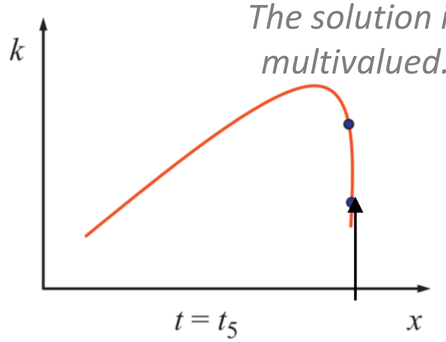



*Compression wave*

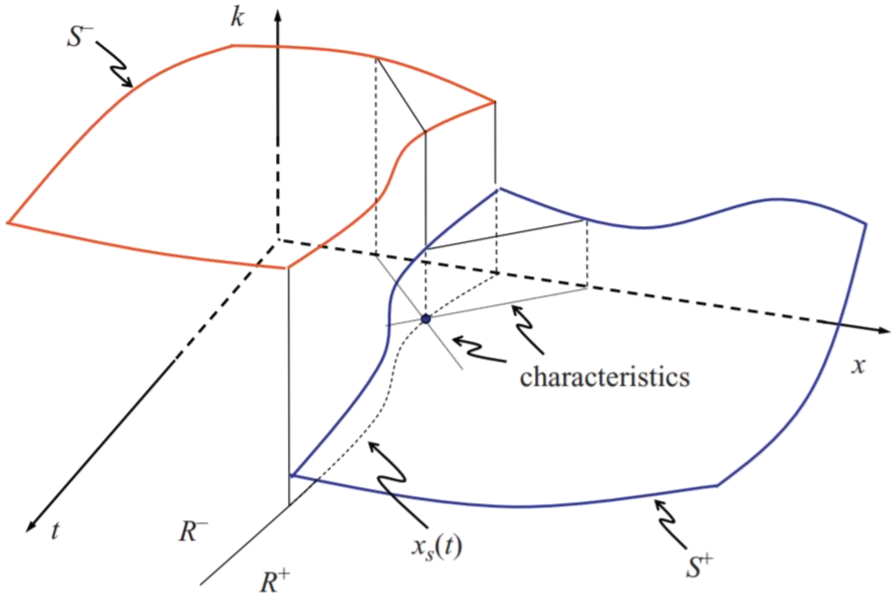



*Expansion wave*

# Shock Waves



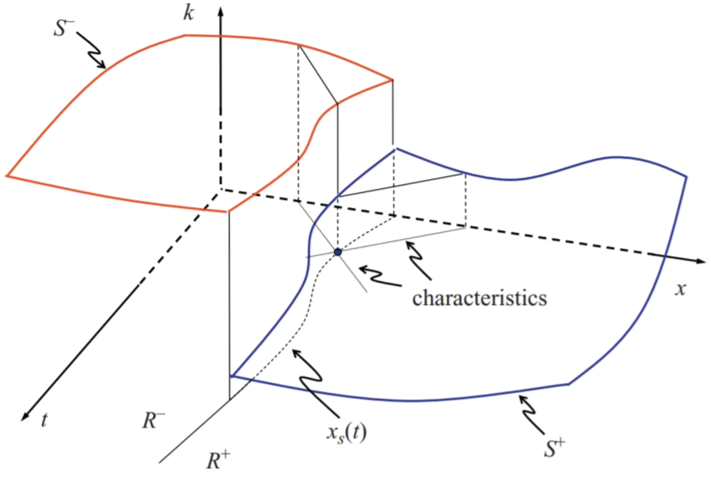

- if two characteristics intersect, the solution at the intersection will be multivalued.
- However, if one allows discontinuity at the intersection, it is possible to construct a piecewise smooth solution.
- Figure illustrates such a solution where curve $x_s\ (t)$ in the $x - t$ plane is a collection of characteristic intersections.
- The solution remains constant along each characteristic and terminates at their intersection.
- Therefore, the curve partitions the solution space into two parts $R^-$ and$R^+$ and, consequently, separates the solution into two smooth pieces $S^-$ and $S^+$.
- The drop or discontinuity of $k$ at the curve denotes an abrupt change of $k$ which creates a *shock wave*.
- Such a piecewise smooth solution of the partial differential equation (PDE) is called a shock wave solution.



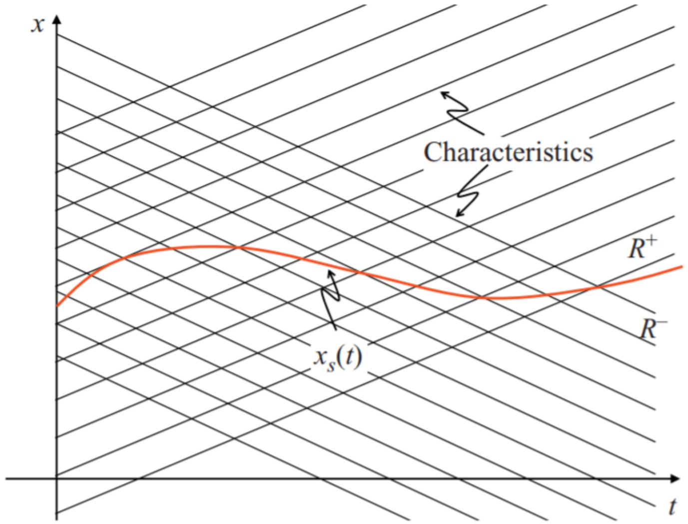

*Piecewise solution—shock wave*

# Shock Waves

- A critical step in the shock wave solution is to find the curve $x_s(t)$ which connects the intersections of characteristics.
- Since the curve represents the locations at which a shock wave forms, such a curve is called a *shock path*.
- In Figure, two families of characteristics are illustrated where a characteristic may have multiple intersections.
- Hence, many curves can be drawn by connecting different sets of intersections and ➔ the shock wave may take different paths.




*Piecewise solution—shock wave*




*Shock path*

# Shock Waves

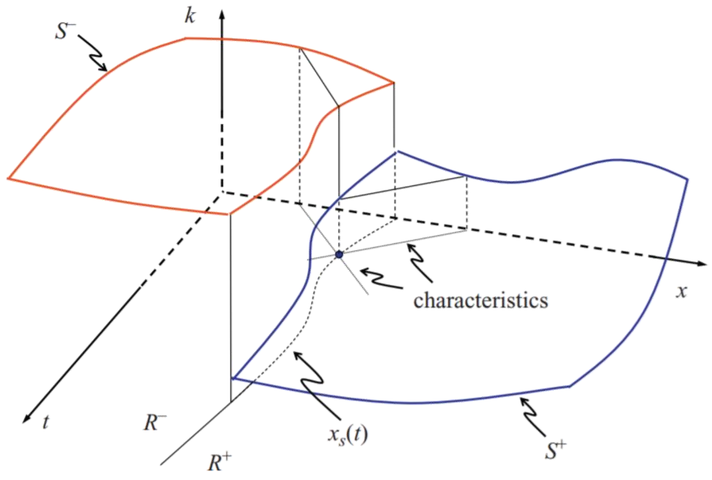
- Fortunately, the underlying conservation law ensures that only one shock path is valid, and such a shock path must satisfy a physical condition called the *Rankine-Hugonoit jump condition*:

$$\frac{dx_s}{dt} = \frac{q(t, x_s^+) - q(t, x_s^-)}{k(t, x_s^+) - k(t, x_s^-)}$$

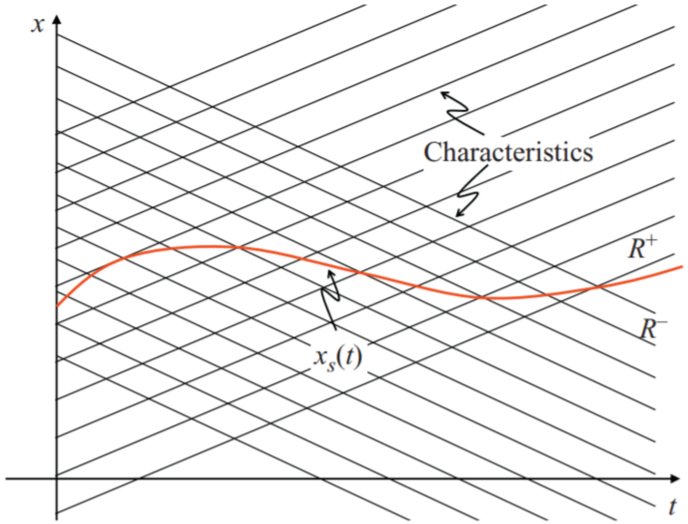
Where :

- $dx_s$ dt is the slope of the shock path,
- $q \; = Q(k)$ as defined in the conservation law,
- $k(t, x_s^- )$ takes the $k$ value on the $R^-$ side,
- $k(t, x_s^+)$ takes the $k$ value on the$R^+$side,
- and similar notation applies to $q(t, x_s^-)$ and $q(t, x_s^+)$.

- Therefore, if one or more intersections on curve $x_s(t)$ are known, one can construct the shock path by starting from the known points and following the slope defined above.



*Piecewise solution—shock wave*



*Shock path*

# Shock Waves

Example

- solve the conservation law with the following initial condition using the shock waves method:

$$\begin{cases} k_t + q_x = 0, \\ q = \frac{1}{2}k^2, \\ k(0,x) = \begin{cases} 1 & \text{if } x \leq 0, \\ 0 & \text{if } x > 0, \end{cases} \\ -\infty < x < \infty, \\ t > 0. \end{cases}$$

# Shock Waves

Problem:

$$\begin{cases} k_t + q_x = 0, \\ q = \frac{1}{2}k^2, \\ k(0,x) = \begin{cases} 1 & \text{if } x \le 0, \\ 0 & \text{if } x > 0, \end{cases} \\ -\infty < x < \infty, \\ t > 0. \end{cases}$$

## Solution

- The slope of the characteristics is $c = dq/dk = k$
- Therefore, characteristics drawn below x = 0 are straight, parallel lines with slope $c = k = 1$ ➔ $q = 1/2k^2 = 1/2$
- Similarly, characteristics drawn above $x = 0$ are horizontal lines with slope $c = 0$. They carry $k = 0$, and hence $q = 0$.
- The origin is a known point on the shock path. According to the *Rankine-Hugonoit jump condition*, the slope of the shock path is:

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = \frac{q(t, x_s^+) - q(t, x_s^-)}{k(t, x_s^+) - k(t, x_s^-)} = \frac{0 - 1/2}{0 - 1} = \frac{1}{2}$$

- Therefore, the shock path is a straight line which starts from the origin with constant slope 1/2—that is:

$$x_s(t) = \frac{1}{2}t$$

# Shock Waves

## Solution

- ➢ Therefore, the solution is: $$k(t,x) = \begin{cases} 1 & \text{if } x \le \frac{1}{2}t \\ 0 & \text{if } x > \frac{1}{2}t \end{cases}$$
- ➢ The solution is illustrated in Figure.
- ➢ Few concepts discussed before are illustrated:
- ➢ A characteristic is a line along which the solution $k$ remains constant;
- ➢ A kinematic wave is a family of straight, parallel characteristics, and a shock wave separates two kinematic waves with an abrupt change of the $k$ value;
- ➢ A shock path is the projection of shock locations onto the $x - t$ plane.

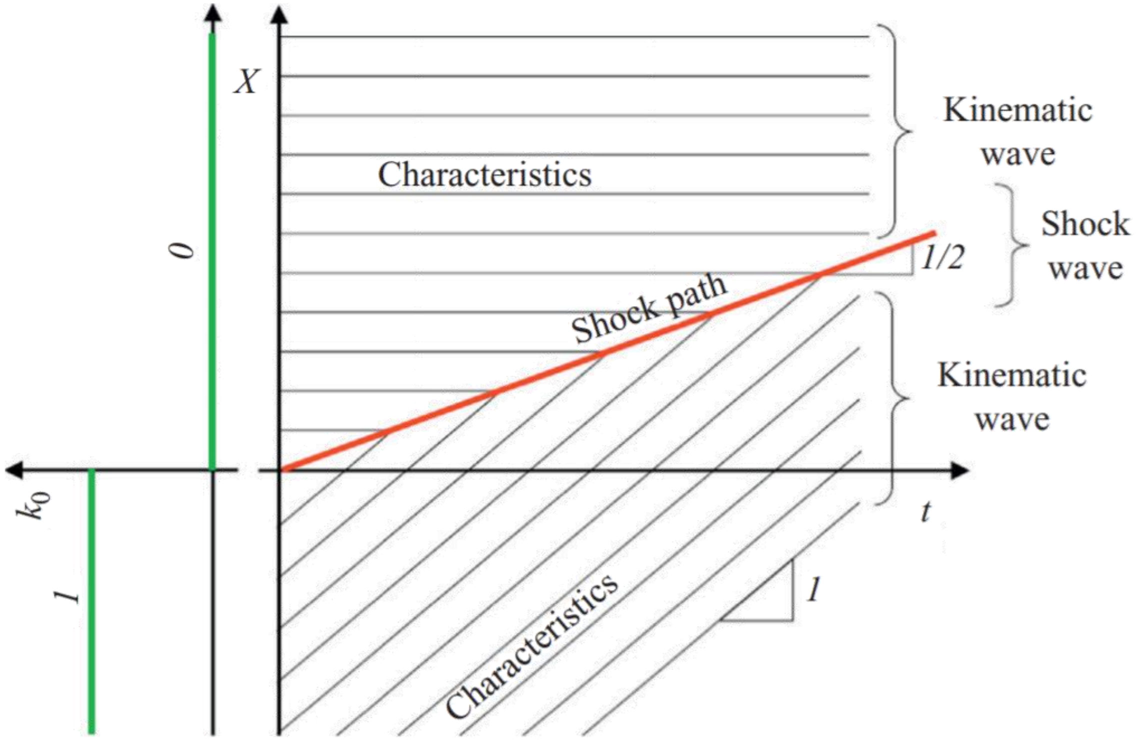
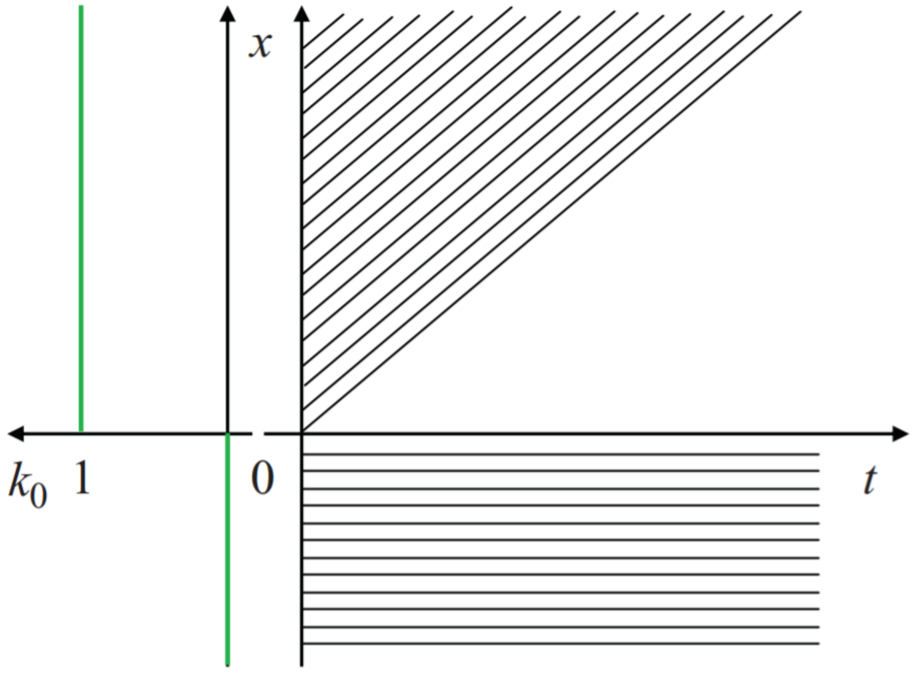



*Illustration of the solution*

# Rarefaction Waves

- If the initial condition in the above example is reversed the two families of characteristics go farther and farther apart, leaving an empty wedge-shaped area in between.
- Characteristics of this PDE are drawn in the Figure.
- Since a characteristic carries a constant k solution, areas swept by characteristics will have solutions.
- An empty area in the solution space means there is no solution in this area.
- To resolve this issue, there should be a means to fill the empty area with characteristics.

$$k(0,x) = k_0 = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x > 0, \end{cases}$$



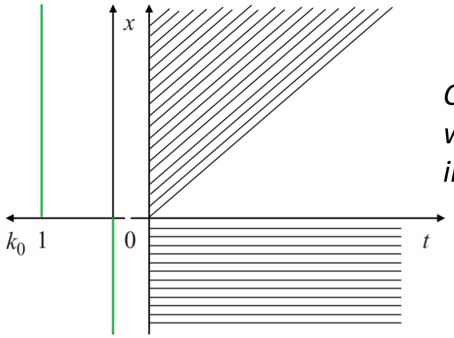

*Characteristics without an intersection*

# Rarefaction Waves

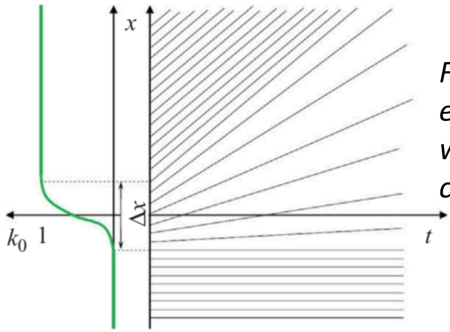
- If one relaxes the step function of the initial condition by assuming that $k0$ varies smoothly from $0$ to $1$ over a small distance $\Delta x$, the slopes of characteristics drawn in $\Delta x$ will gradually increase from $0$ to $1$ so that any point in the solution space is swept by one and only one characteristic.

- To return to the step function of the initial condition, one takes the limit $\Delta x \rightarrow\ 0$.

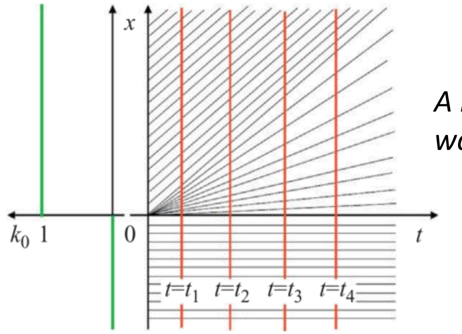
- Now the empty area is filled with a fan of characteristics drawn from the origin. If one cuts the solution space with a few planes $t\ =\ t_0, t_1, t_2$, …, with $t_0$ passing the origin and other planes at consequently later times, one obtains a time development of the solution as shown in Figure 7.11. Notice that the profile of the solution is thinned out or rarefied as time moves on. Hence, this fan of characteristics represents a rarefaction wave.




*Characteristics without an intersection*




*Filling an empty area with characteristics*




*A rarefaction wave*

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ЯСЕНЕВО
Thank
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