Fundamental diagrams

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Introduction

- It is reasonable to assume that drivers will on average make the same decisions under the same average conditions.

- If drivers are driving in a traffic flow that has a certain speed u, they will on average remain the same distance headway s with respect to the preceding vehicle.

- This implies that if we would consider a stationary traffic flow, it is reasonable to assume that there exists some relation between
  - the traffic density k and the mean speed v,
  - and between the density and the flow, or the flow and the speed.

- This relation is sometimes referred to as the equilibrium relation and is expressed as:

$$q = k \times v$$

Introduction

- Analogy to fluid dynamics: The traffic flow is similar to the flow of fluids and the traffic state is described based on speed v, density k and flow q.

- It depends on
  - The different properties of the road (width of the lanes, grade),
  - The composition of the flow (percentage of trucks, fraction of commuters, experienced drivers, etc.),
  - External conditions (weather and ambient conditions),
  - Traffic regulations,
  - etc.

- This chapter introduces the concept of the fundamental diagram and the relationship between macroscopic characteristics of traffic.

Introduction

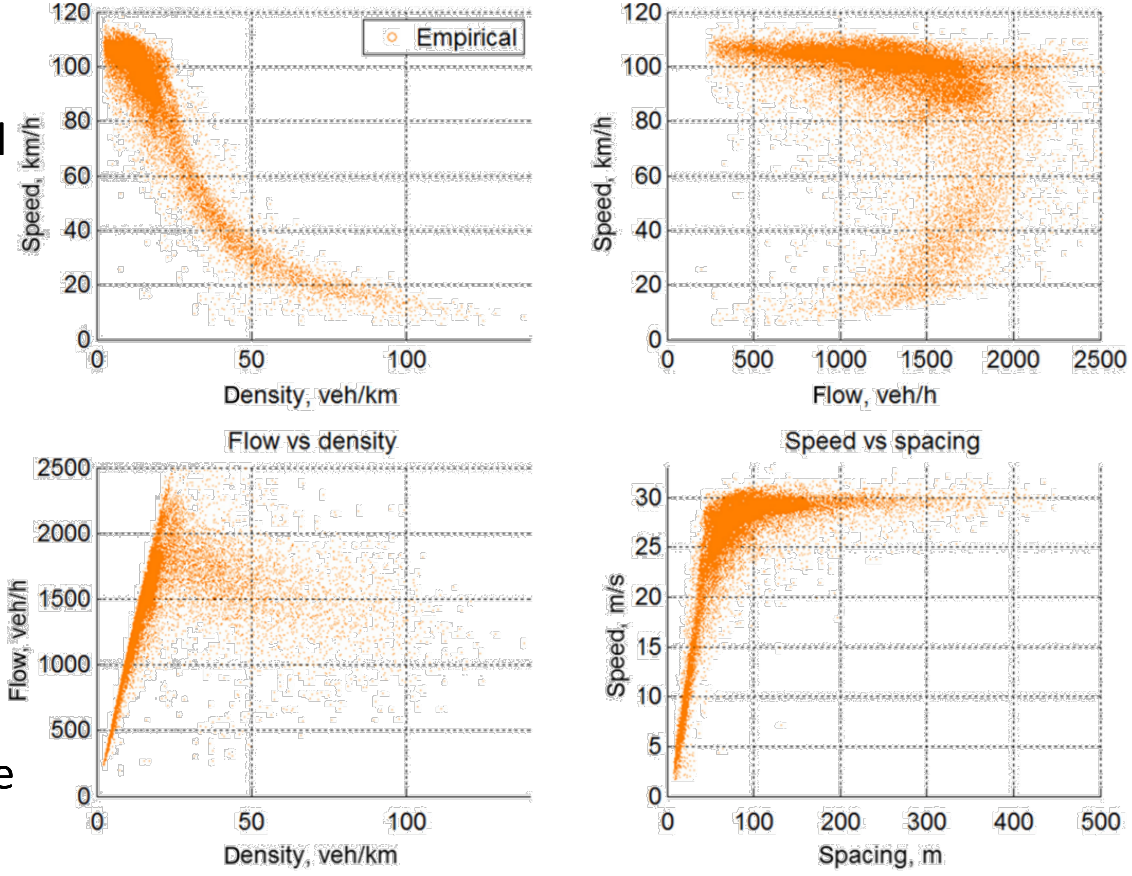
- Image illustrates the data captured by a video camera in Georgia NaviGAtor, Georgia's intelligent transportation system.
- As discussed before, traffic data can be extracted from video images by means of image processing.
- A virtual detector has been placed in each lane.
- Collected data:
  - Aggregated volume over 20s periods
  - Occupancy
  - Speed
  - Average vehicle length
  - Density
  - Gap

NAVIGATOR

An image captured by video cameras from NaviGAtor data collection system

Introduction

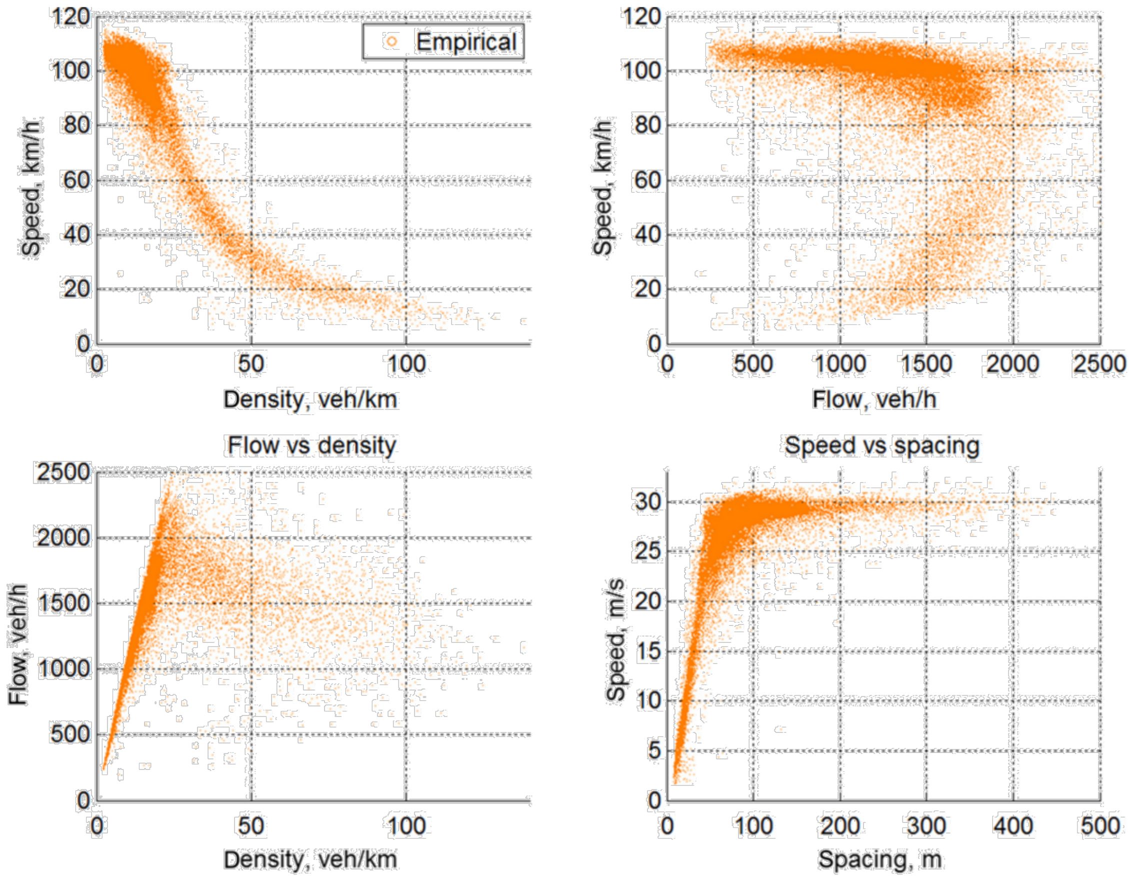
- The plots are generated with use of traffic data collected at a fixed location.
- One-year worth of field observations aggregated to 5 min (i.e., each point in the figure represents the traffic condition observed in 5 min).
- The traffic speed here is the time mean speed since it is impossible to calculate the space mean speed from aggregated point sensor data.
- Density is estimated from flow and speed (The exact method is not described).
- Therefore, such plots are location specific—that is, plots generated from different locations may differ.
- Time information is lost in the figure ➔ One could not deduce the time when a data point was observed.
- As such, the figure actually depicts an equilibrium or steady-state relationship ➔ models of such a relationship without a reference to time are termed "equilibrium models" or "steady-state models."



Observed q-k-v relationships from NaviGAtor data collection system

Introduction

- Each plot exhibits a trend which suggests a pairwise relationship among flow, speed, and density.
- For example, the top-left plot reveals a decreasing relationship between speed and density with two intercepts intuitively known.
- One intercept represents a scenario where there are very few vehicles on the road (i.e., $k \to 0$). Hence, one may drive at the desired speed without being blocked by a slow driver ($v \to v_f$, *the free-flow speed*).
- The other intercept corresponds to a scenario where everyone rushes home. As such, the road is jammed ($k \to k_j$, *the jam density*), resulting in a stop-and-go condition ($v \to 0$).
- Numerous functions have been proposed to establish the relationship between speed and density.



Observed q-k-v relationships from NaviGAtor data collection system

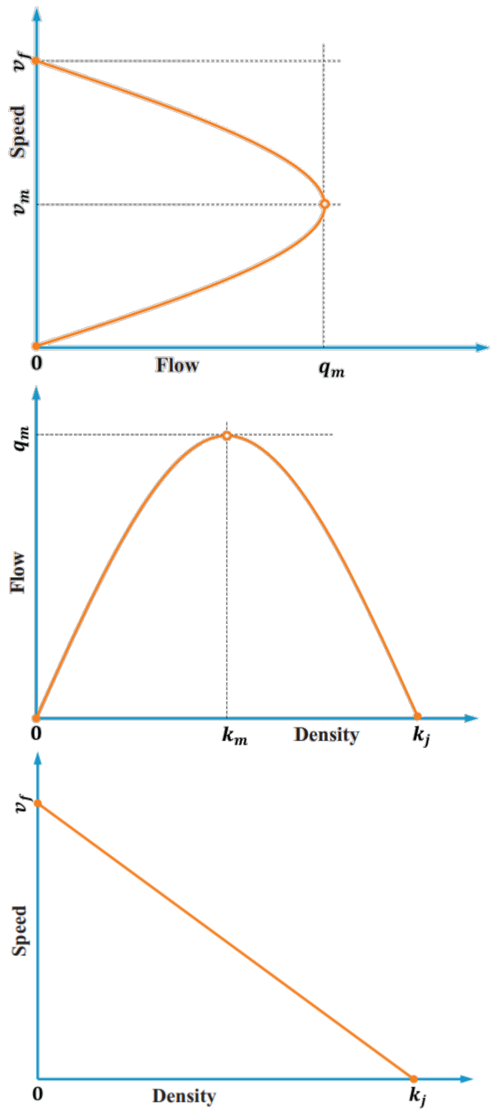
The Greenshields Model

- Greenshields proposed the use of a linear function to summarize the speed-density relationship.
- First fundamental diagram to describe traffic flow conditions.
- The Greenshields function can be completely determined from knowledge of two points on the line:
  - $k = 0, v = v_f$
  - $k = k_j, v = 0$
- The speed-density $v - k$ relationship can be expressed as:

$$v = v_f(1 - \frac{k}{k_j})$$

- Combining the identity $q = k \times v$ and eliminating v, one can derive the flow-density $q - k$ relationship implied by the Greenshields model:

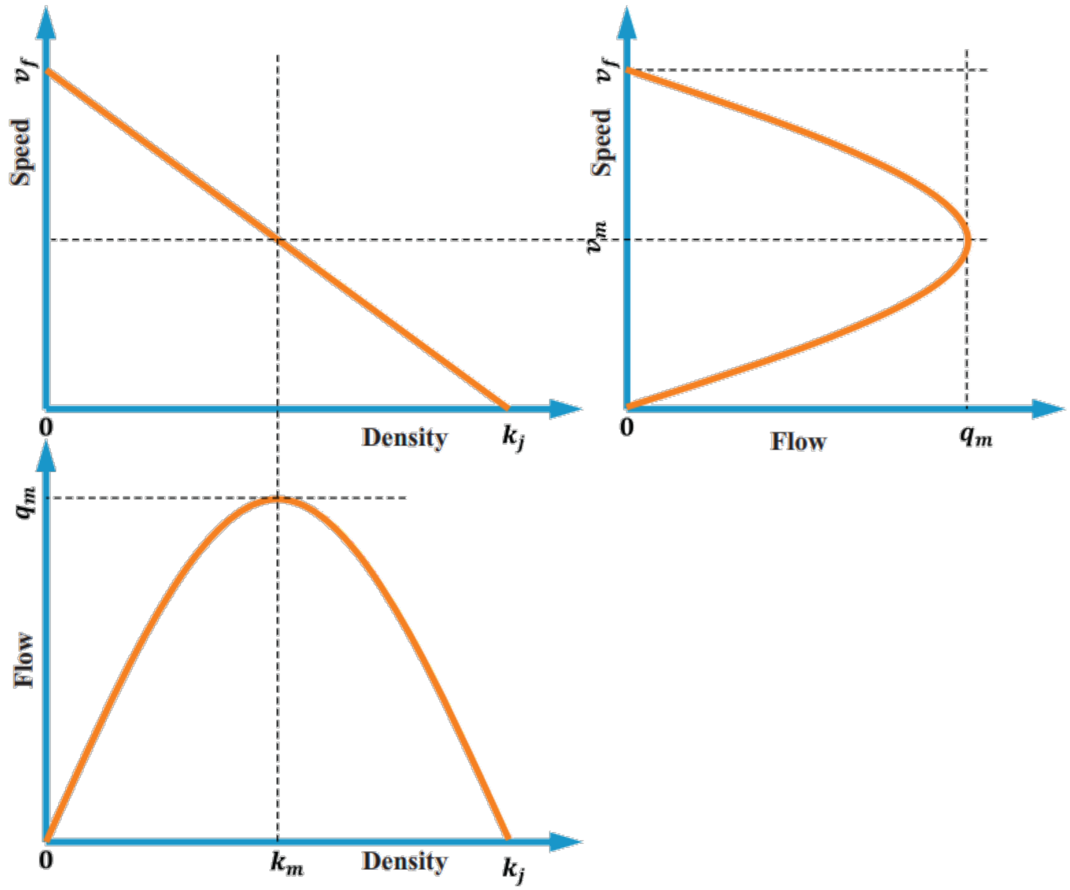
$$q = v_f(k - \frac{k^2}{k_j})$$



The Greenshields model

Special points of the fundamental diagram

- It is interesting to note a few special points on the curve.
- When the density is close to zero ($k \to 0$), the flow drops to zero ($q \to 0$) since the road is almost empty
- When the road is jammed ($k = k_j$), the flow also becomes zero ($q = 0$) because no one can move.
- Starting from the origin ($k = 0, q = 0$), flow increases as density increases.
- This trend continues until, at some point ($k = k_m$), the flow peaks ($q = q_m = \frac{v_f k_j}{4}$).
- After this point, the flow begins to drop as the density continues to increase, and the flow becomes zero ($q = 0$) when the density reaches the jam density ($k = k_j$).
- In this notation, q_m is the maximum flow – that is, the capacity – and k_m is the optimal (critical) density – that is, the density when the flow peaks.



The Greenshields model

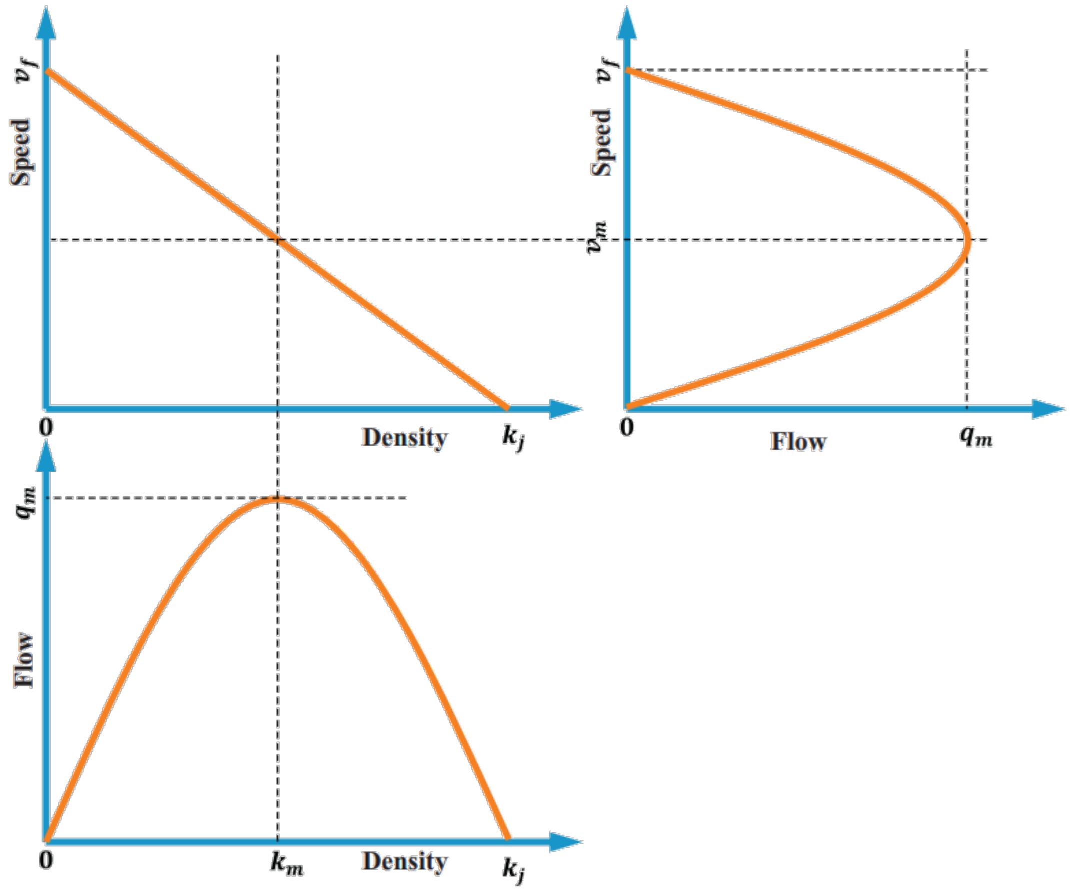
Special points of the fundamental diagram

- The Greenshields model

$$q = k_j\left(v - \frac{v^2}{v_f}\right)$$

- When the flow is close zero ($q \rightarrow 0$), two scenarios are possible:
  1. The road is nearly empty and the few vehicles on the road can move at freeflow speed ($v \rightarrow v_f$);
  2. The road is jammed, so that no one can move ($v \rightarrow 0$).

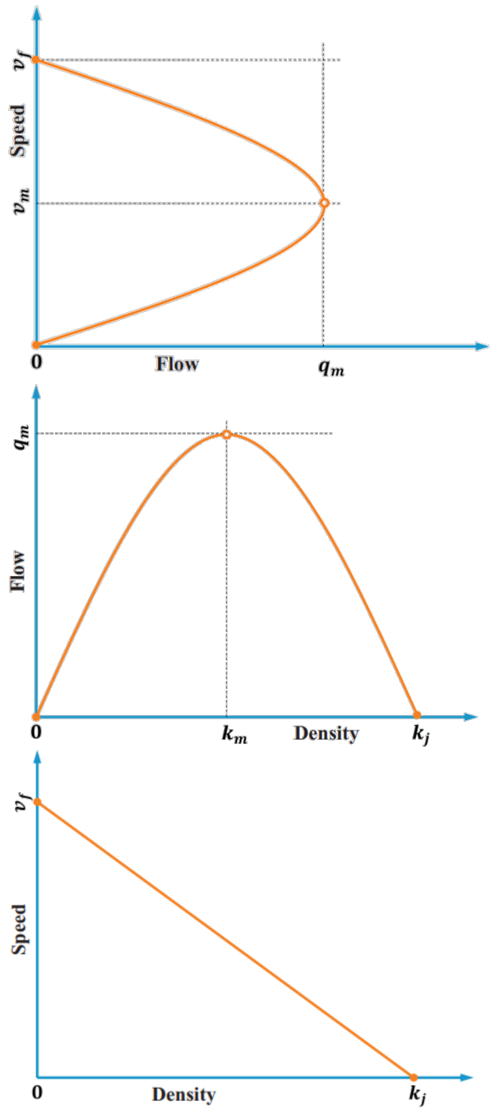
- Entering a given flow value less than the capacity into the equation will normally result in two speeds:
  1. A lower one, which corresponds to a worse traffic condition.
  2. A higher one, corresponding to a better traffic condition. When the flow reaches capacity ($q = q_m$), the two speeds become one, which is called the optimal (critical) speed, v_m.



The Greenshields model

Special points of the fundamental diagram

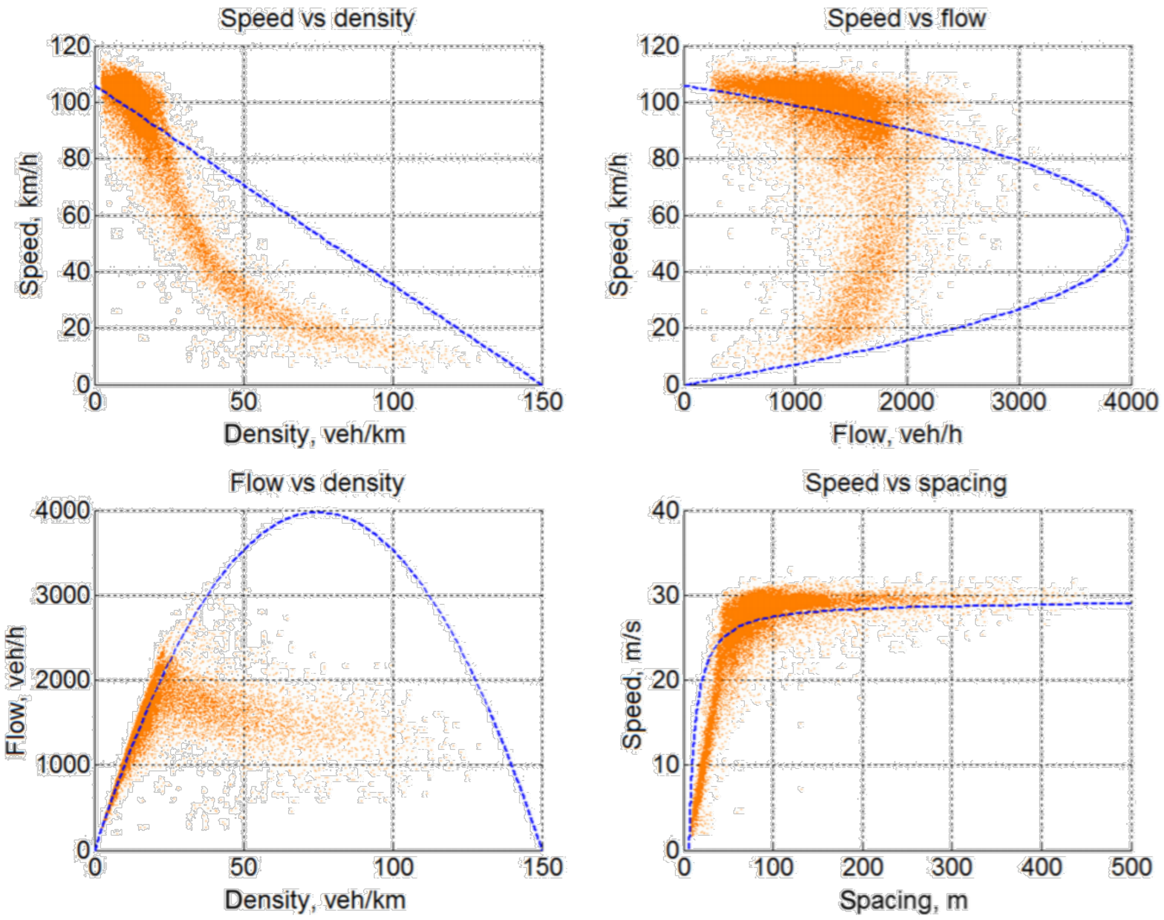
- Note that the three pairwise relationships – that is, the speed-density, flow-density, and speed-flow relationships – reflect different facets of the flow-speed-density relationship.
- Hence, they have different applications in traffic flow theory. For example:
  - The speed-density relationship relates a driver's speed choice to the concentration of vehicles around the driver. Therefore, the relationship is typically used in traffic flow theory to understand how drivers adjust their speeds in response to traffic in their vicinity – that is, modeling drivers' car-following behavior.
  - As will be seen later, the flow-density relationship is convenient for explaining the propagation of disturbances in traffic flow (such as waves and their velocities) and, hence, is frequently used in dynamic traffic flow modeling.
  - Anyone who is familiar with highway capacity and level of service (LOS) will immediately recognize that the speed-flow relationship is extensively used by traffic engineers to perform highway capacity analysis and determine the LOS on freeways and multilane highways.



The Greenshields model

Single-regime models

- Greenshields model is simple and elegantly depicts the relation between speed v, density k and flow q.
- Empirical observations reveal that the model suffers from a lack of accuracy.
- For example, the model predicts that the capacity (q_m) occurs at half the jam density $(k_m = \ 1/2k_j)$.
- If an average vehicle length of 6 m is assumed, the jam density would be somewhere around $1000/6 \ \approx \ 164$ veh/km. Half of this number is 82 veh/km.
- However, field observations suggest that k_m is most likely in the range of $25 - 40 \ veh/km$.
- Also, unlike the way that speed decreases linearly with density, field observations show that free-flow speed can be sustained up to a density of about 15 veh/km before a noticeable speed drop can be observed.



The Greenshields model and empirical data

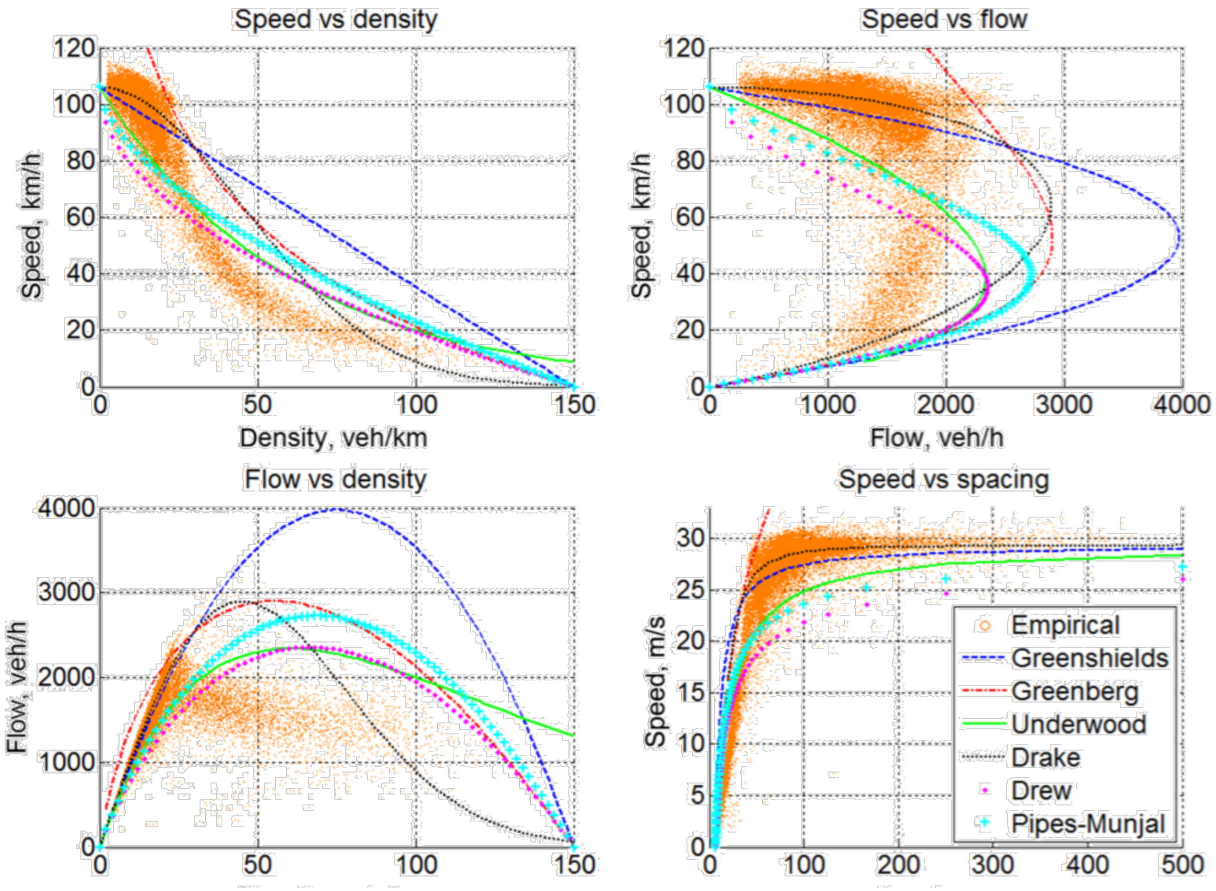
Single-regime models

- Inspired by Greenshields's pioneering work, many models were proposed subsequently to formulate speed-density relationships with various degrees of fitting quality.

Single-regime models

Authors	Model	Parameters
Greenshields [9]	$v = v_f\left(1 - \frac{k}{k_j}\right)$	v_f, k_j
Greenberg [10]	$v = v_m \ln\left(\frac{k_j}{k}\right)$	v_m, k_j
Underwood [11]	$v = v_f e^{-\frac{k}{k_m}}$	v_f, k_m
Drake et al. [12]	$v = v_f e^{-\frac{1}{2}\left(\frac{k}{k_m}\right)^2}$	v_f, k_m
Drew [13]	$v = v_f\left[1 - \left(\frac{k}{k_j}\right)^{n+\frac{1}{2}}\right]$	v_f, k_j, n
Pipes [14] and Munjal [15]	$v = v_f\left[1 - \left(\frac{k}{k_j}\right)^{n}\right]$	v_f, k_j, n

v_f is free-flow speed, k_j is jam density, v_m is optimal speed, k_m is optimal density, and n is an exponent.



Single-regime models and empirical data

Single-regime models

- All these models are one equation models, meaning that the models apply to the entire range of density. Hence, these models are called single-regime models.

Single-regime models

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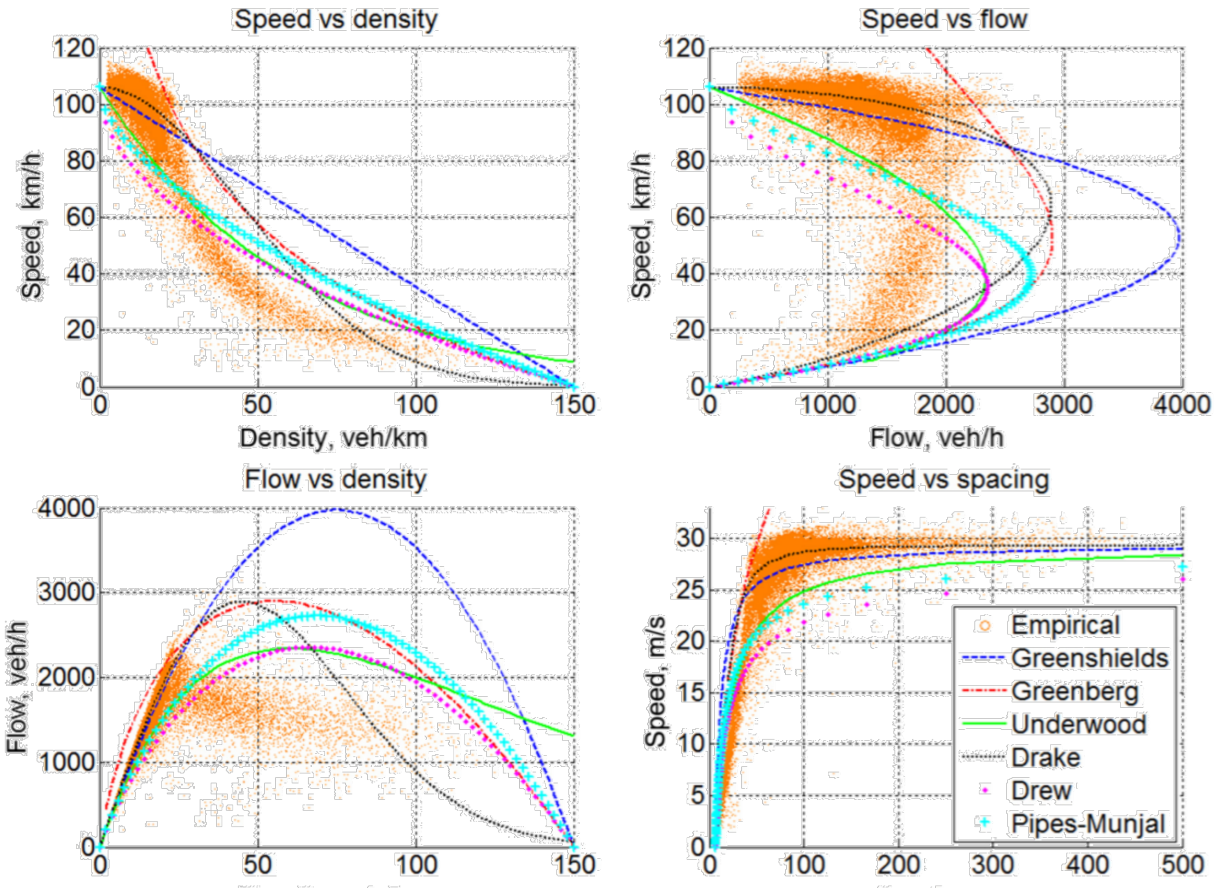

Single-regime models and empirical data

Single-regime models

- Single-regime models such as the Greenshields model, the Greenberg Model, etc. are typically simple because they involve few parameters.

- In addition, in these models, the derivatives of flow with respect to density dq/dk exist at each point in the entire range of density. This makes these models mathematically appealing because dq/dk can be very useful later in dynamic macroscopic modeling such as in solving the LWR model.

- Moreover, these macroscopic models are closely related to a family of microscopic car-following models.

- These models typically suffer from poor fitting quality.

Multi-regime models

- It seems that none of these single-regime models are able to fit the empirical observations reasonably well over the entire density range.
  - Some models are good in one density range, while others are superior in another range.
- The inability of single-regime models to perform well over the entire range of density prompted researchers to think about fitting the data in a piecewise manner using multiple equations.
- This gave rise to multi-regime models.
- They fit better to empirical data compared to single-regime models, but their piecewise formulation makes them less attractive.

Multi-regime models

Regimes models	Free flow	Transitional	Congested
Edie model	$v = 108e^{-k/163.9}$ $k \leq 20$	- -	$v = 47\ln(162.5/k)$ $k > 20$
Two-regime model	$v = 108 - 0.515k$ $k \leq 30$	- -	$50 - 0.33k$ $k > 30$
Modified Greenberg model	$v = 103$ $k \leq 20$	- -	$v = 52\ln(150/k)$ $k > 20$
Three-regime model	$v = 108 - 0.5k$ $k \leq 20$	$v = 120 - 1.5k$ $20 < k \leq 65$	$v = 40 - 0.256k$ $k > 65$

The Greenshields model and empirical data

Multi-regime models

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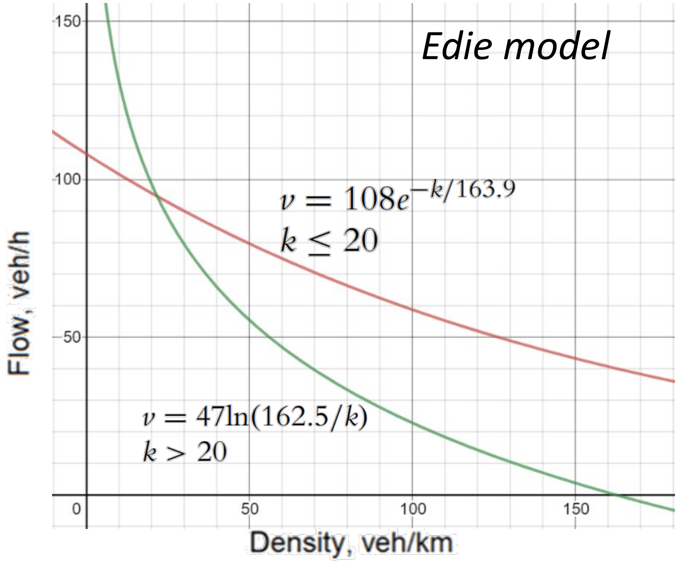
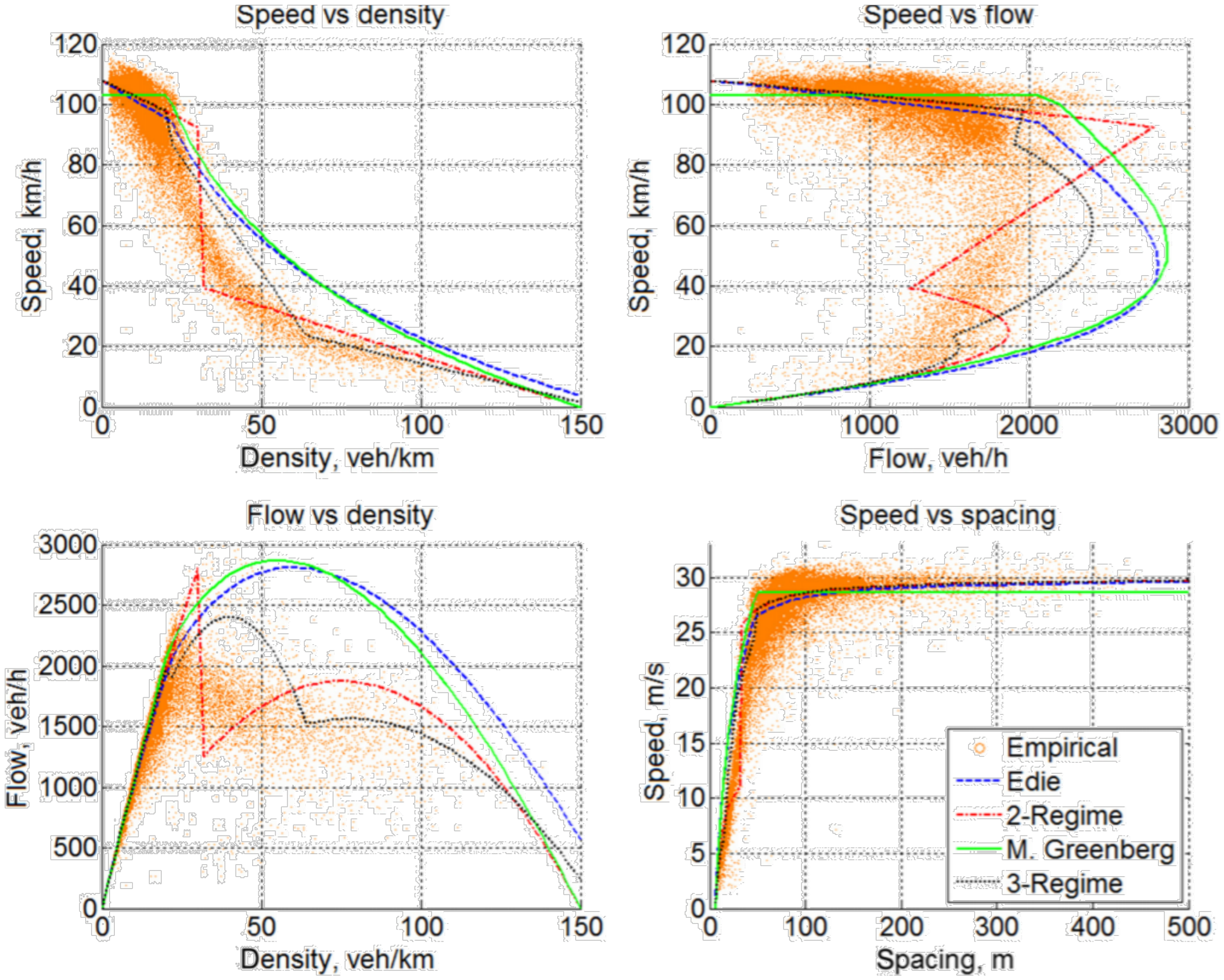


Source: https://www.desmos.com/calculator



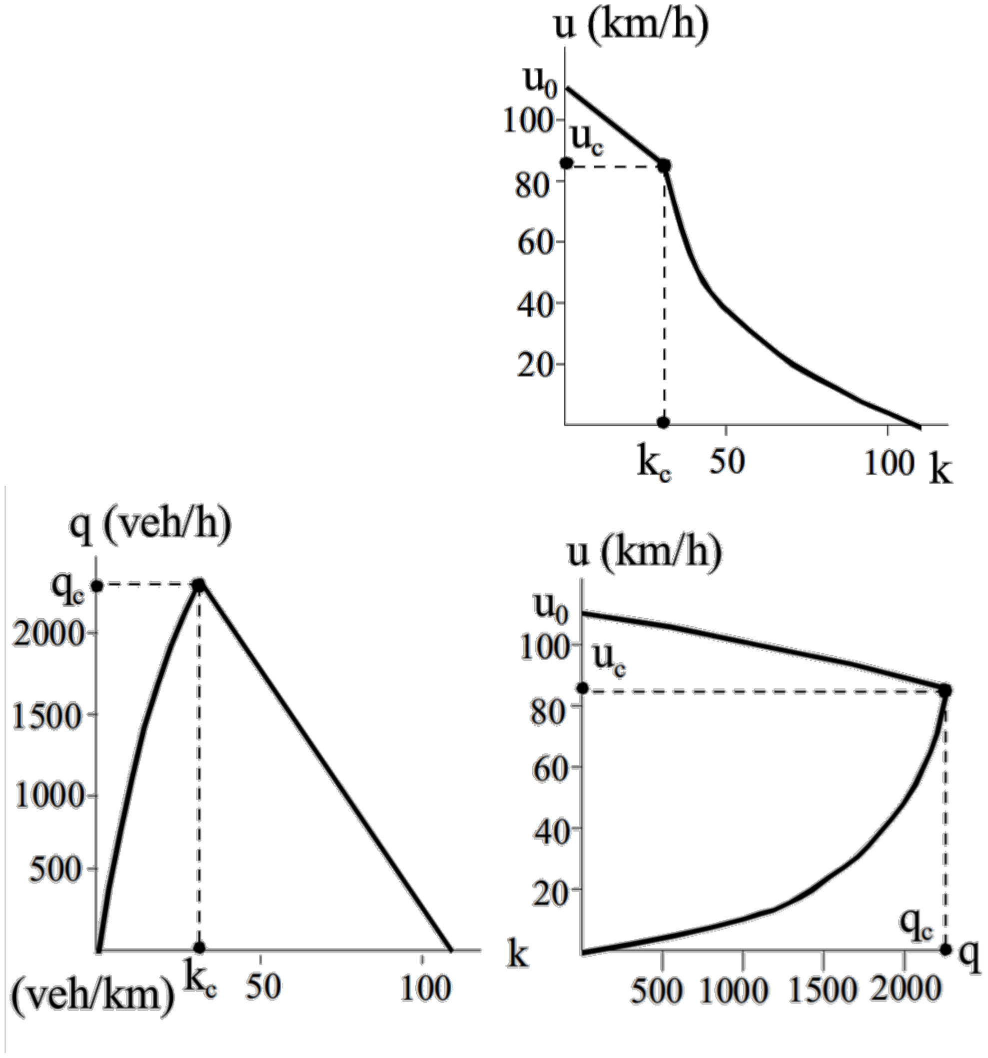
Multi-regime models and empirical data

Multi-regime models

- Smulders introduced the following equations:

$$u(k) = \begin{cases} u_0\,(1 - k/k_j) & \text{for } k < k_c \\ \gamma\,(1/k - 1/k_j) & \text{for } k > k_c \end{cases}$$

  - u_0 is the free-flow speed,
  - k_j is the jam density,
  - k_c is the critical density (i.e. k_m).
- γ follows from the requirement that $u(k)$ is continuous at point $k = k_c$ ➔ $\gamma = u_0 k_c$



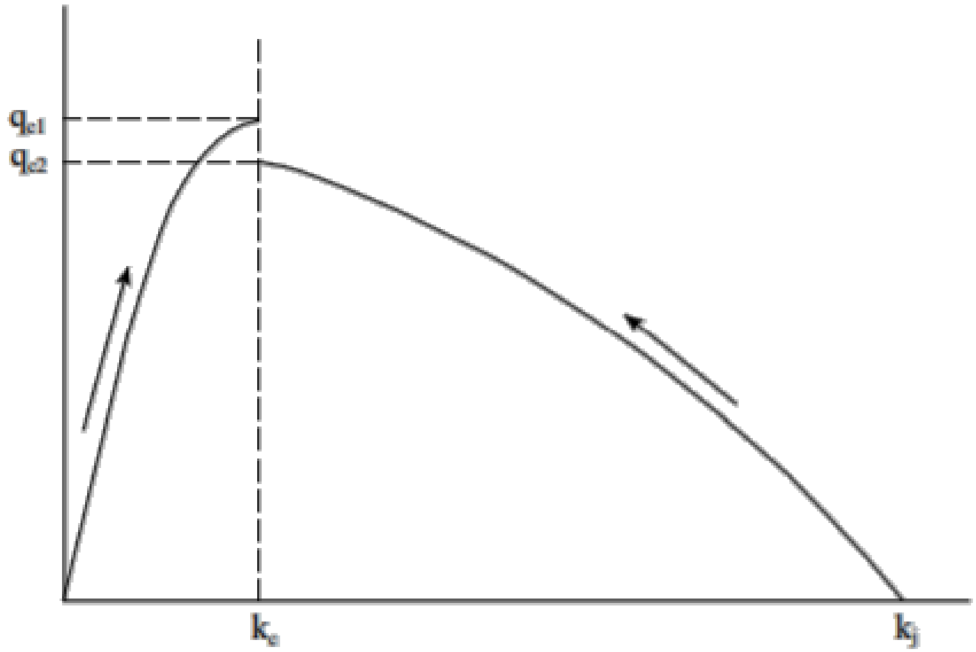
Smulders' fundamental diagram

Concept of discontinuous diagram

- Edie was the first researcher to indicate the possibility of a discontinuity in the diagram around the capacity point.

- This idea is based on the observation that a traffic stream with increasing density (starting from stable or free flow) reaches a higher capacity value (‘free flow capacity’, q_{c1}) than a traffic stream starting from a congested state (in the extreme case from a standing queue) that ends in the so called ‘queue discharge capacity’, q_{c2}.

- This idea also called ‘capacity drop’.



Fundamental diagram with discontinuity

Wu's fundamental diagram with capacity drop

- Wu has developed a model for the diagram with a capacity drop, based on assumptions about microscopic behaviour.
- In this model two regimes are distinguished: free flow and congested flow.
- Free flow has densities from $k = 0$ up to $k = k_1$. Congested flow has a density range from $k = k_2$ up to the jam density k_j. Both regimes are overlapping in terms of density range, i.e. $k_1 > k_2$.

Main assumption for free flow state

- In this state it is assumed the traffic flow is a mixture of free driving vehicles with mean speed u_0 and platoons with speed u_p . If the fraction of free driving vehicles (in terms of density) is p_{free}, then the fraction of vehicles in platoon is $(1 - p_{free})$, and the overall space mean speed equals:

$$u = p_{free}u_0 + (1 - p_{free})u_p$$

- $p_{free} = 1 \Rightarrow u = u_0$ for $k = 0$ and $p_{free} = 0 \Rightarrow u = u_p$ for $k = k_1$

Wu's fundamental diagram with capacity drop

- If we suppose (it can be also be argued based on queueing theory) that for a two-lane roadway (for one-directional traffic) p_{free} decreases linearly with density:

$$p_{free} = 1 - k/k_1$$

- For a three-lane roadway: $p_{free} = 1 - (k/k_1)^2$

- In general the exponent equals the number of lanes minus 1, i.e. for n lanes

$$u(k) = (1 - (k/k_1)^{n-1})u_0 + (k/k_1)^{n-1}u_p$$

- This function implies that $u(k)$ starts flatter if the roadway has more lanes.
- At density k_1 every vehicle drives in a platoon with speed u_p. This implies a relation between the net time headway, h^f_{nett}, in this platoon, the effective vehicle length $(1/k_j)$ and speed.

$$k_1 = \left(u_p h^f_{nett} + \frac{1}{k_j}\right)^{-1}$$

- Given parameters u_p and k_j , either h^f_{nett} or k_1 is a free parameter of the model.
- Wu has chosen h^f_{nett} because it can be observed in practice more easily than k_1, especially if it is assumed that h^f_{nett} is a constant parameter for all free flow states.

Wu's fundamental diagram with capacity drop

Assumption for congested flow state

- In congested flow every vehicle is (more or less) in a car-following state and maintains a constant net time headway, h^c_{nett}, over the density range $k_2 < k < k_j$

- This assumption implies a straight line for the congested part of the function $q(k)$. Hence in this aspect the model is the same as Daganzo's and Smulders'.

$$\left.\begin{aligned} s_{nett} &= u h^c_{nett} \quad \text{with } u = \text{mean speed} \\ s_{gross} &= 1/k_j + u h^c_{nett} \\ k &= \frac{1}{s_{gross}} = \frac{1}{1/k_j + u h^c_{nett}} \end{aligned}\right\} u = \frac{1}{h^c_{nett}}\left(\frac{1}{k} - \frac{1}{k_j}\right) \quad \text{and apply } q = ku \rightarrow q = \frac{1}{h^c_{nett}}\left(1 - \frac{k}{k_j}\right)$$

- Parameter k_2 : This parameter can be determined by assuming that the maximum speed of the congested state is at most equal to the speed that corresponds to 100 percent platooning at free flow state:

$$k_2 = \frac{1}{1/k_j + u_p h^c_{nett}} = \left(u_p h^c_{nett} + \frac{1}{k_j}\right)^{-1}$$

Wu's fundamental diagram with capacity drop

- Example: Typical values for a two-lane roadway with 100 % cars are:
  - $u_0 =$ free flow speed $= 110\ km/h$;
  - $up =$ speed of free flow platoon $= 80\ km/h$;
  - $k_j =$ jam density $= 150\ veh/km$;
  - $h_{nett}^{f} =$ net time headway in free flow platoon $= 1.2\ s$;
  - and, $h_{nett}^{c} =$ net time headway at congested flow $= 1.6\ s$.

- Because the net time headway at free flow is smaller than at congestion the two branches of the fundamental diagram in the $u - k$ plane do not coincide.
  In the example the free flow capacity is $2400\ veh/h$ and the discharge capacity $1895\ veh/h$ (21 % less).
- Capacity drops found in practice are usually smaller.

<u>Note:</u> The capacity drop offers a substantial possible benefit of ramp metering. If one can control the input flows such that the flow on the freeway does stay below, say, $2200\ veh/h\ per\ lane$, then most of the time the smaller discharge capacity is not relevant.

The State-of-the-Art Models

- Further research emphasizes single-regime models, which are mostly coupled with the development of microscopic car-following models.
- Details of these car-following models and their associated equilibrium models will be discussed later in this course.

Newell Nonlinear Model

- The Newell nonlinear model involves three parameters and takes the following form:

$$v = v_{\mathrm{f}} \left(1 - \mathrm{e}^{-\frac{\lambda}{v_{\mathrm{f}}}\left(\frac{1}{k} - \frac{1}{k_j}\right)} \right)$$

- v_f is the free-flow speed,
- k_j is the jam density,
- λ is the slope of the speed-spacing curve.

The State-of-the-Art Models

Del Castillo and Benítez Model

- Also involving three parameters, the model of del Castillo and Benítez takes the following form:

$$v = v_f \left(1 - e^{1-e^{\frac{|C_j|}{v_f}\left(\frac{k_j}{k}-1\right)}} \right)$$

  - v_f is the free-flow speed,
  - k_j is the jam density,
  - C_j is the kinematic wave speed at the jam density.

The State-of-the-Art Models

Van Aerde Model

- The Van Aerde model involves four parameters and takes the following form:

$$k = \frac{1}{c_1 + c_3 v + c_2/(v_\mathrm{f} - v)}$$

$$c_1 = \frac{v_\mathrm{f}}{k_\mathrm{j} v_\mathrm{m}^2}(2v_\mathrm{m} - v_\mathrm{f})$$

$$c_2 = \frac{v_\mathrm{f}}{k_\mathrm{j} v_\mathrm{m}^2}(v_\mathrm{f} - v_\mathrm{m})^2$$

$$c_3 = \frac{1}{q_\mathrm{m}} - \frac{v_\mathrm{f}}{k_\mathrm{j} v_\mathrm{m}^2}$$

  - v_f is the free-flow speed,
  - v_m is the optimal speed,
  - q_m is the capacity,
  - k_j is the jam density.

The State-of-the-Art Models

Intelligent Driver Model

- The intelligent driver model involves four parameters and takes the following form:

$$k = \frac{1}{(s_0 + vT)[1 - (\frac{v}{v_f})^\delta]^{-1/2}}$$

- v_f is the free-flow speed,
- s_0 is the jam distance,
- T is the safe time headway,
- δ is the acceleration exponent.

The State-of-the-Art Models

Longitudinal Control Model

- The longitudinal control model involves four parameters and takes the following form:

$$k = \frac{1}{(\gamma v^2 + \tau v + l)[1 - \ln(1 - \frac{v}{v_f})]}$$

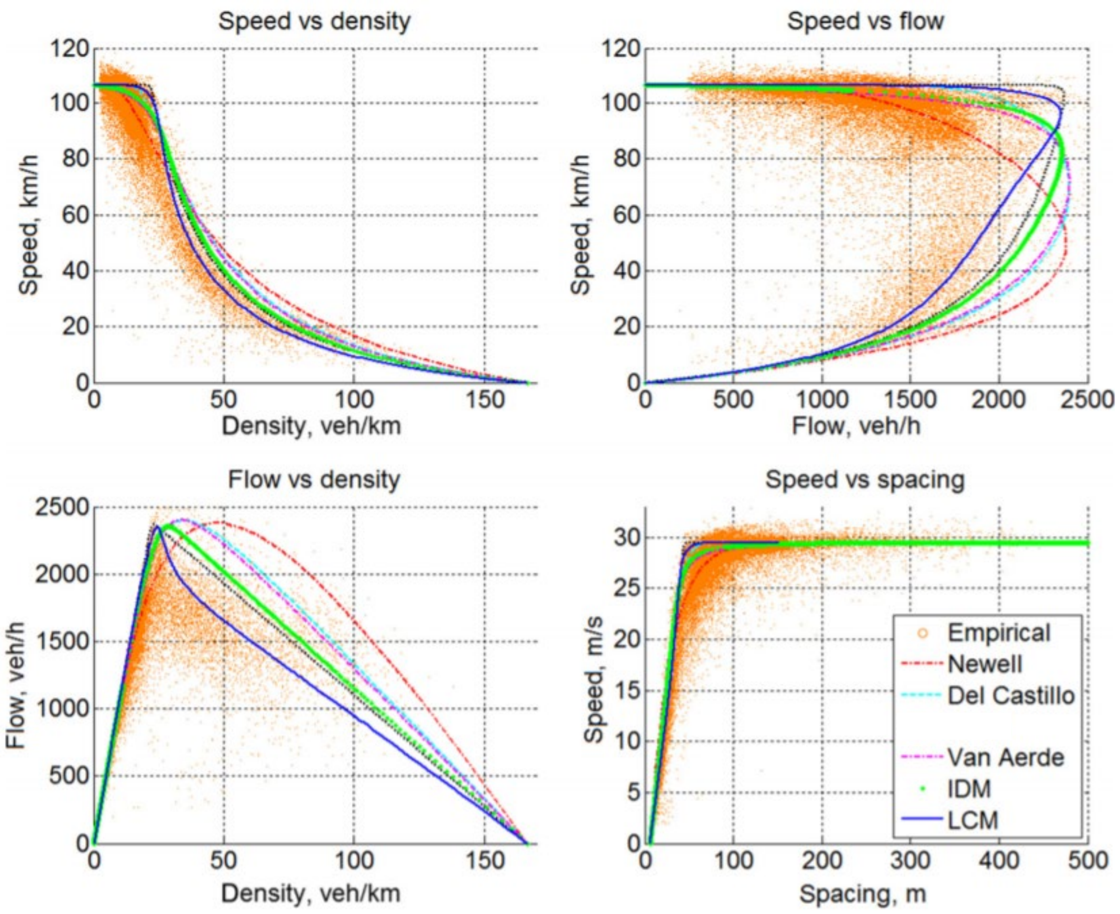
- v_f is the free-flow speed,
- $l = \frac{1}{k_j}$ is the nominal vehicle length, which is the reciprocal of the jam density k_j,
- τ is the perception-reaction time,
- γ is the aggressiveness.

The State-of-the-Art Models

- To illustrate their features, the above models are fitted to empirical data.
- The following general principles apply when one is fitting the models:
  1. Fix the free-flow speed v_f of all the models to roughly the same value observed in the data,
  2. Fix the jam density k_j of all the models to roughly the same value observed in the data,
  3. Fix the capacity to roughly the same value observed in the data by tweaking the remaining parameters

Model parameters

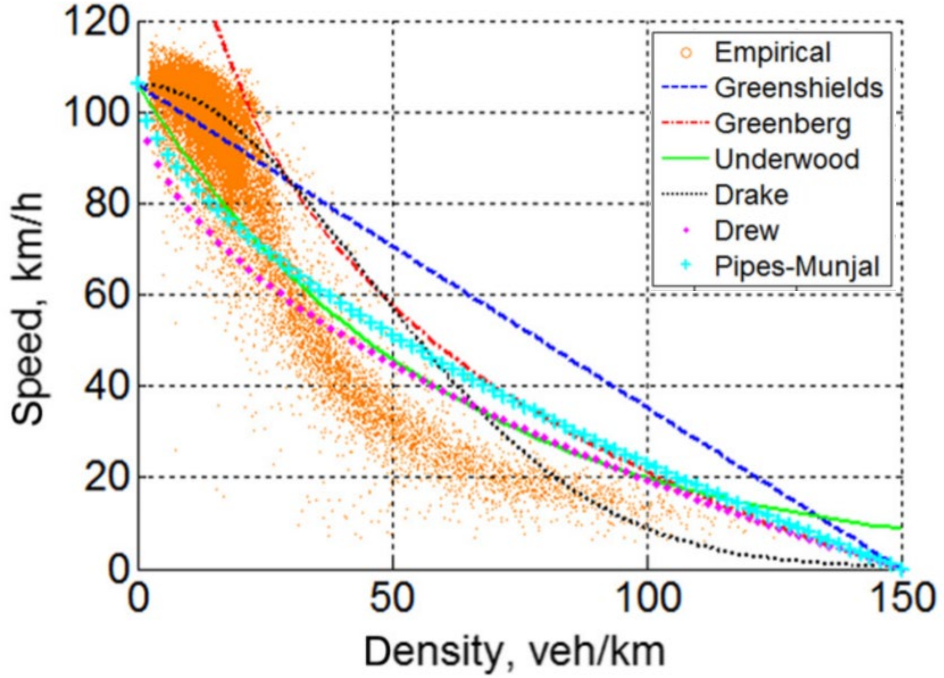
Models	Parameters
Newell model	v_f = 106 km/h; k_j = 167 vehicles/km; λ = 1.25 1/s
Del Castillo and Benítez model	v_f = 106 km/h; k_j = 167 vehicles/km; C_j = 20 km/h
Van Aerde model	v_f = 106 km/h; k_j = 167 vehicles/km; v_m = 20 km/h; q_m= 2400 veh/h
Intelligent driver model	v_f = 106 km/h; s_0 = 6 m; T = 1.25 s; δ = 15
Longitudinal control model	v_f = 106 km/h; l = 6 m; τ = 1.3 s; γ = -0.04 s^2/m



State-of-the-art models fitted to empirical data

Stochastic Models

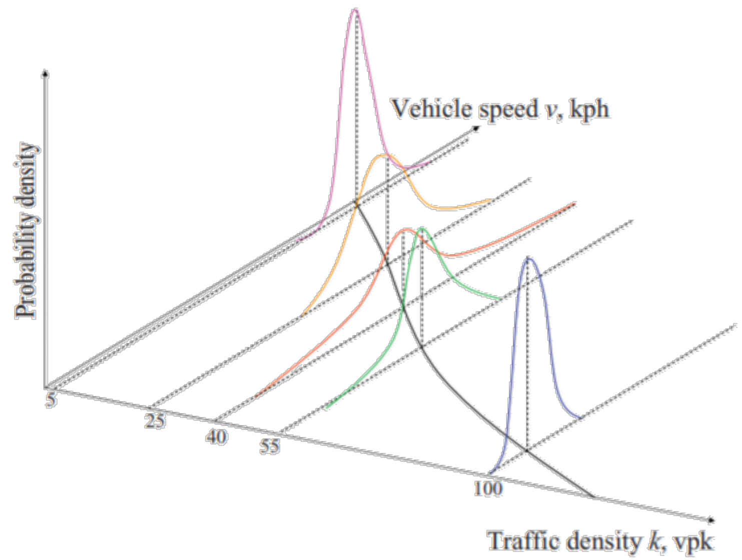
- Though all relationships presented above take deterministic forms, the actual relationships are essentially quite stochastic.
- For example, a speed-density relationship may predict that when the density k is $12\ veh/km$, the speed v will be $96\ km/h$.
- However, in reality, the observed speed may vary over a certain range, forming a distribution.
- Figure illustrate the scattering effect of empirical observations and how deterministic models fail to capture such an effect.



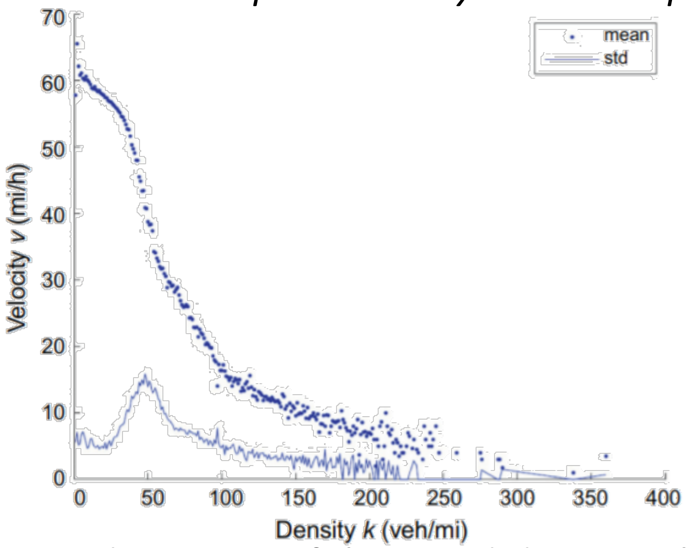
Deterministic Speed-Density relationship

Stochastic Models

- A step forward in modeling the speed-density relationship is to consider the scattering effect by representing speed as a distribution at each density level.
- Empirical observations seem to support such a proposition.
- For example, the observed mean and standard deviation of the speed-density relationship are plotted in the figure below.
- Hence, the deterministic speed-density relationship in the form $v = f(k)$ may be replaced by the following one in generic form: $v = f\big(k, \omega(k)\big)$,
  - where ω is a distribution parameter dependent (at least) on density k.
- In this model, since speed is a distribution at each density level, the model is essentially a stochastic one.



Three-dimensional representation of the speed-density relationship

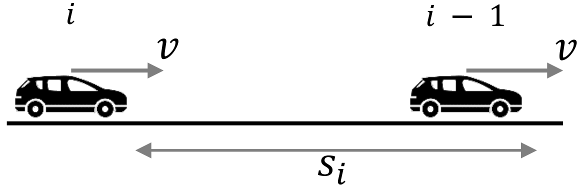


Mean and variance of the speed-density relationship

Fundamental diagram based on a car-following model

- Consider a situation in which vehicle i drives behind vehicle $i - 1$. Vehicle i considers a gross distance headway of s_i. Both vehicles have the same speed v. Driver i includes the following in determination of the gross distance headway:

    1. Vehicle $i - 1$ may suddenly brake and come to a complete stop,
    2. Driver i has a reaction time of T_r seconds,
    3. Braking is possible with a deceleration of a (with $a > 0$),
    4. When coming to a full stop behind the preceding vehicle, the net distance headway between the vehicles $i - 1$ and i is at least d_0,
    5. The deceleration of the vehicle $i - 1$ is α times the deceleration of vehicle i.



- It should be clear that the parameter α is somehow a measure for the aggressiveness of the driver: larger values of α imply that the follower assumes more abrupt deceleration of the follower, which will result in larger distance headways.

Fundamental diagram based on a car-following model

- To calculate the minimal distance s_i to the leader of vehicle i:

$$s_i + v_i^2 - \frac{1}{2\alpha a_i} = L + d_0 + v_i T_r + \frac{v_i^2}{2a_i}$$

The gross distance headway + braking distance of the leader = length of the follower + security margin + reaction distance + braking distance follower

- If we assume that $s_0 = L + d_0$
(gross stopping distance headway = vehicle length + safety distance margin)
then we have:

$$s_i = s_0 + v_i T_r + \frac{v_i^2}{2a_i}\left(1 - \frac{1}{\alpha}\right)$$

Fundamental diagram based on a car-following model

- To transform this microscopic relation into a macroscopic description of traffic flow, we need to make the following substitutions:
  - Replace the gross distance headway s_i by the inverse of the density $1/k$,
  - Replace the individual speed v_i by the flow speed u,
  - Replace the gross stopping distance headway s_0 by the inverse of the jam density $1/k_j$.

- We get

$$\frac{1}{k} = \frac{1}{k_j} + uT_r + \frac{u^2}{2a}\left(1 - \frac{1}{\alpha}\right)$$

- Since $q = ku$, we get the following expression for the relation between flow and speed:

$$q(u) = \frac{u}{\frac{1}{k}} = \frac{u}{\frac{1}{k_j} + uT_r + \frac{u^2}{2a}\left(1 - \frac{1}{\alpha}\right)}$$

Fundamental diagram based on a car-following model

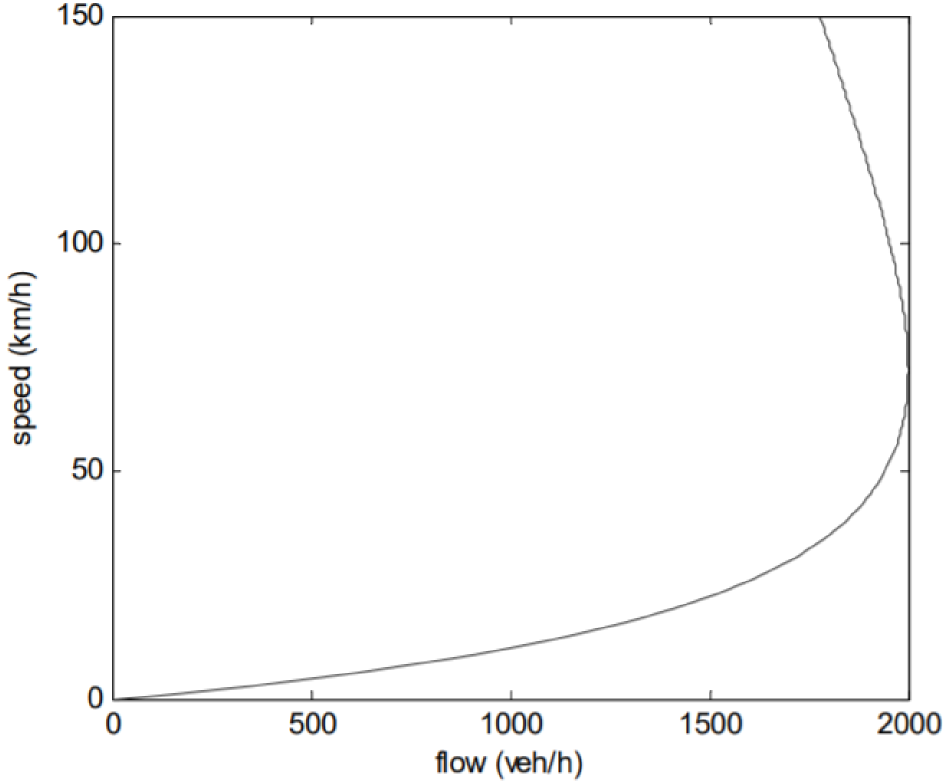
- The $u - q$ relation is depicted
- Note that the speed is not restricted in this model

- We can determine the speed u_c for which capacity results by taking the derivative of q with respect to u and setting it to zero:

$$u_c = \sqrt{\frac{2}{k_j}\frac{a}{(1-1/\alpha)}}$$

$$q_c = \frac{u_c k_j}{2 + T_r u_c k_j}$$

$$k_c = \frac{k_j}{2 + T_r u_c k_j}$$



Flow - speed curve derived from simple car-following model

Fundamental diagram based on a car-following model

- We can also express q_c as a function of α, T_r,k_j and determine the effect of changes in the parameters on the capacity:

$$q_c = \frac{1}{T_r + \sqrt{\frac{2}{k_j}\frac{1-1/\alpha}{a}}} < \frac{1}{T_r}$$

  - The reaction time T_r $\downarrow$
  - The jam-density k_j $\uparrow$
  - The braking deceleration a $\uparrow$ (improved braking system)
  - The parameter α is closer to 1 (drivers are less cautious)

  Road capacity $\uparrow$

- It turns out that as time goes by, capacity increases steadily.
- For example, in the USA the capacity has increases from $2000\ veh/h$ (in 1950) to $2400\ veh/h$ (in 2000) under ideal circumstances.
- Using the derived model, we can (partially) explain this 20% increase.
- We can thus explain the capacity increase by a decrease in α (drivers are more daring and are more experienced), a shorter reaction time T_r and higher deceleration capacity a.

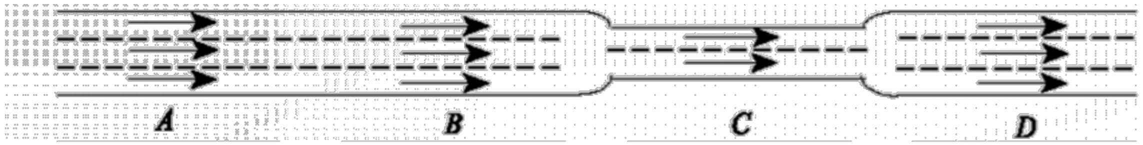
General points

- If one wants to determine the fundamental diagram for a road section the following points are relevant:
- Does one need the complete diagram or only a part of it; e.g. only the free operation part $(k < kc)$ or only the congestion branch. A more fundamental point is whether it is possible to determine the complete diagram at one cross-section.
- Is the road section homogeneous? If this is the case, one can do with observations at a single cross-section. Otherwise road characteristics are variable over the section and a method such as the moving observer might be suitable.
- Period of analysis: If this is chosen too short, random fluctuations will have too much influence; if it is too long then it is questionable whether the state of the flow is stationary over the period. In practice the balance between randomness and stationarity has led to periods of 5 to 15 minutes.
- Finally, one has to estimate the parameters of the model chosen. This is mostly done by using a regression technique. A given set of data points can often be used to fit quite a few different models. In general models without too many parameters are preferred.

Data collection location

- To obtain a representative fundamental diagram one has to carry out measurements at various sites and during different periods.
- This will be illustrated with the data one can get when taking measurement around an overloaded bottle-neck.
- This phenomena is depicted for a roadway of 3 lanes with a bottle-neck (b-n) section of 2 lanes wide.

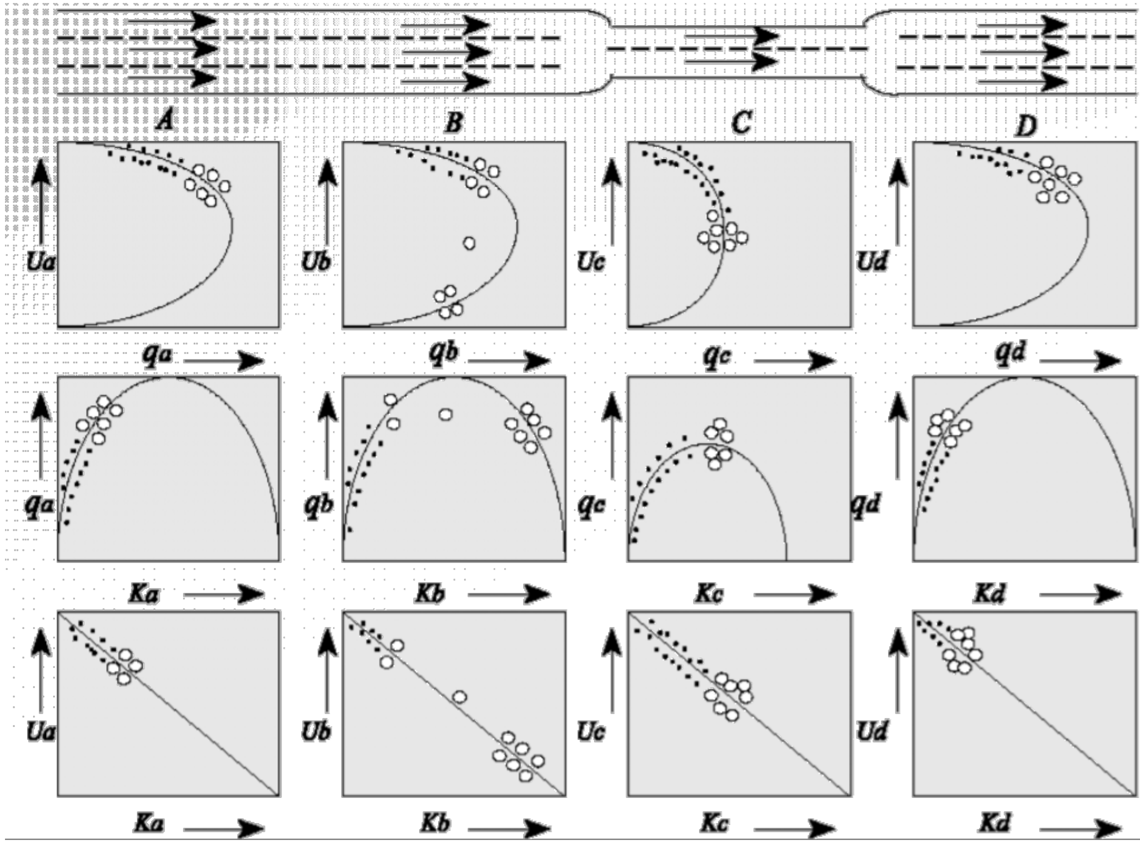
- Measurements will be carried out at 4 cross-sections:
  A. This cross section is so far upstream that congestion due to an overloading of the b-n will not reach it.
  B. This cross-section is closer to the b-n and congestion will reach it.
  C. A cross-section inside the b-n.
  D. A cross-section downstream of the b-n.



Traffic flow conditions at different cross-sections for an under and oversaturated bottle-neck

Data collection location

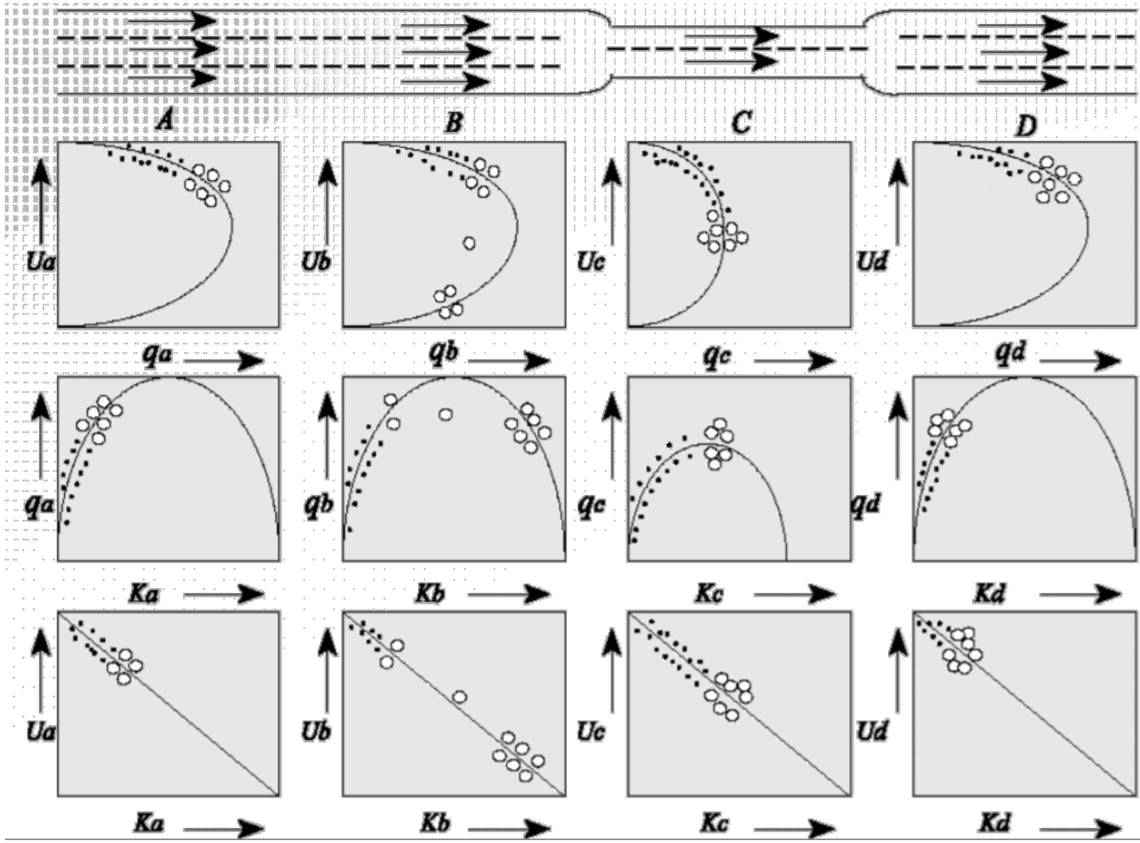
- The fundamental diagrams of Greenshields have been assumed to hold for all cross-sections.
- They are the same for cross-sections A, B and D. For cross-section C the form is similar, but the capacity and jam density are $2/3$ of the values at A.
- We assume that the intensity increases gradually from a low value to a value that is just a little smaller than the capacity of the b-n; this capacity is $2C_0$ with $C_0 = \ capacity\ of\ one\ lane$. The data points resulting from such a demand pattern are depicted as $*$ in the diagrams.
- At cross-section A, B and D intensity is not higher than $2/3$ of the capacity. This means free operations with high speeds.
- At cross-section C the capacity is nearly reached.
- Now we assume demand increases further until a value that equals 2.5 times C_0 and discuss which data points we will get on the 4 cross-sections.



Traffic flow conditions at different cross-sections for an under and oversaturated bottle-neck

Data collection location

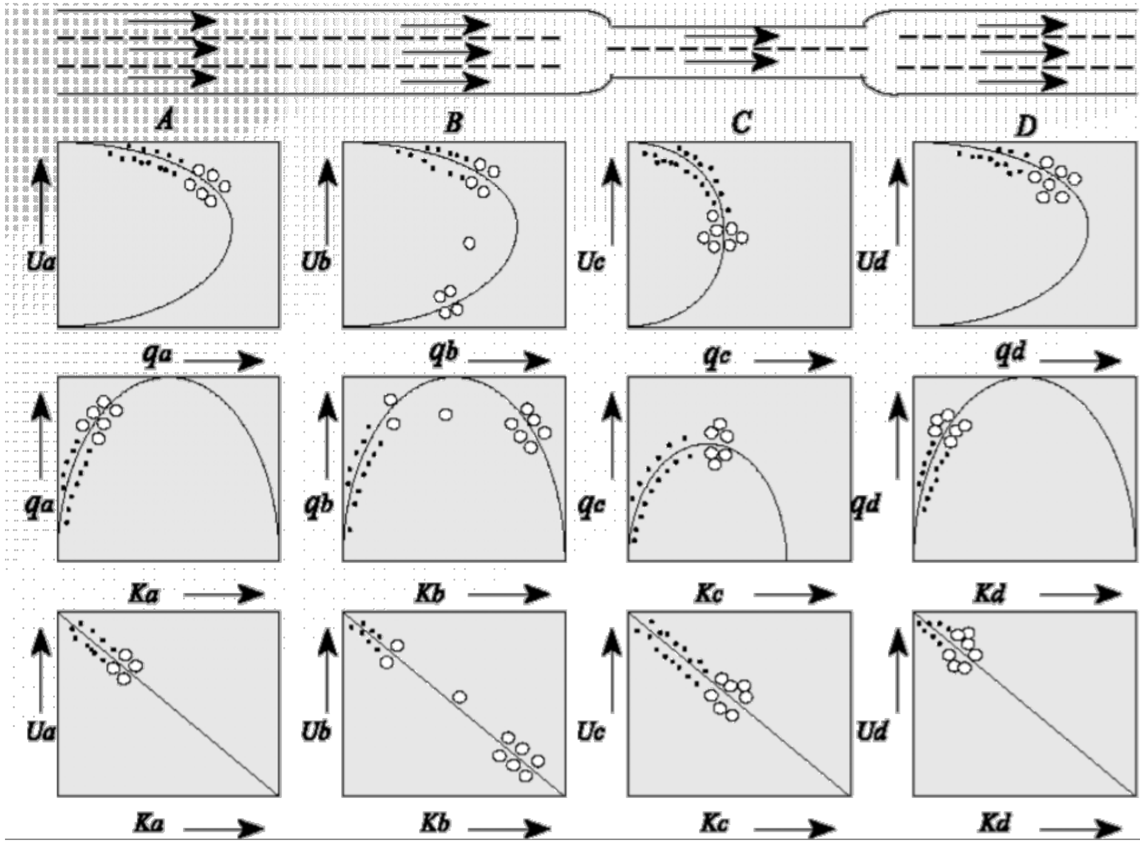
- $\boldsymbol{A}$: It has been assumed that congestion will not reach this cross-section, so the state of the flow remains free. The data points, depicted by open circles, are in a range of 4000 to 5000 vehicles per hour and speeds remain high.
- $\boldsymbol{B}$: When the higher demand reaches the beginning of the b-n, congestion will start, move upstream and reach cross-section B after some time. Before that moment data points are still on the free flow part of the diagram and afterwards on the congestion part. The flow then equals (on average) the capacity of the b-n and the mean speed equals the speed corresponding to this flow according to the congested part of the diagram.
- $\boldsymbol{C}$: In the b-n, flow is limited to the capacity value of the b-n.
- $\boldsymbol{D}$: Here the flow is not larger than $2C_0$ because the b-n does not let through more vehicles and traffic operation remains free.



Traffic flow conditions at different cross-sections for an under and oversaturated bottle-neck

Data collection location

- When demand is reduced to low values the process will develop in reverse order.
- Looking to the total results it appears that at cross-section A and D only free traffic operation can be observed.
- In the b-n one can observe the free flow part of the diagram.
- Most information about the diagram is seen at cross-section B.
- It should be realized that also here the information about the congested part of the diagram is rather limited. To collect data about the complete congested part of the diagram, one requires a b-n with a capacity varying from zero to, in this case, $3C_0$.
- The site of data collection determines which traffic flow states one can observe.
- Only in a bottle-neck one can observe a long-lasting capacity state. If one wants to estimate capacity and has observations carried out at sites not being a bottle-neck, some form of extrapolation is always required.

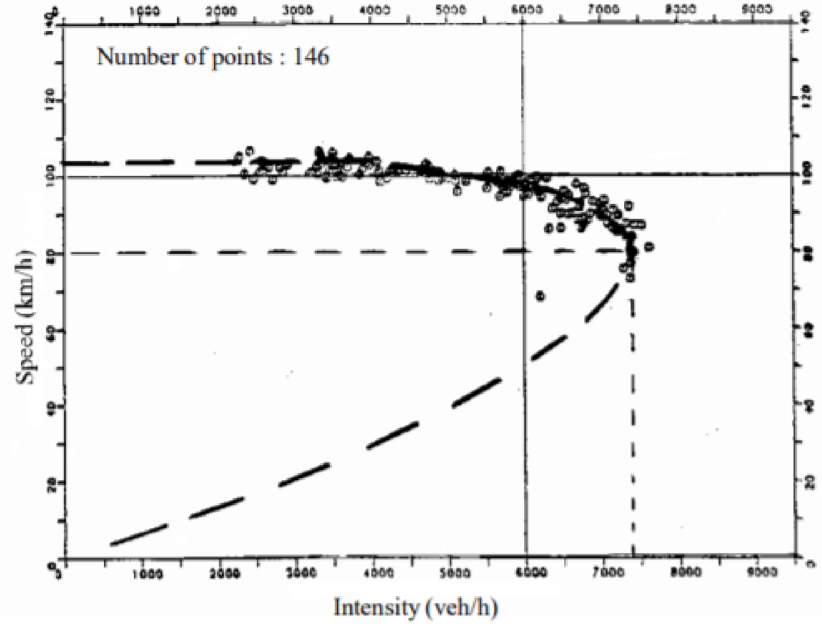


Traffic flow conditions at different cross-sections for an under and oversaturated bottle-neck

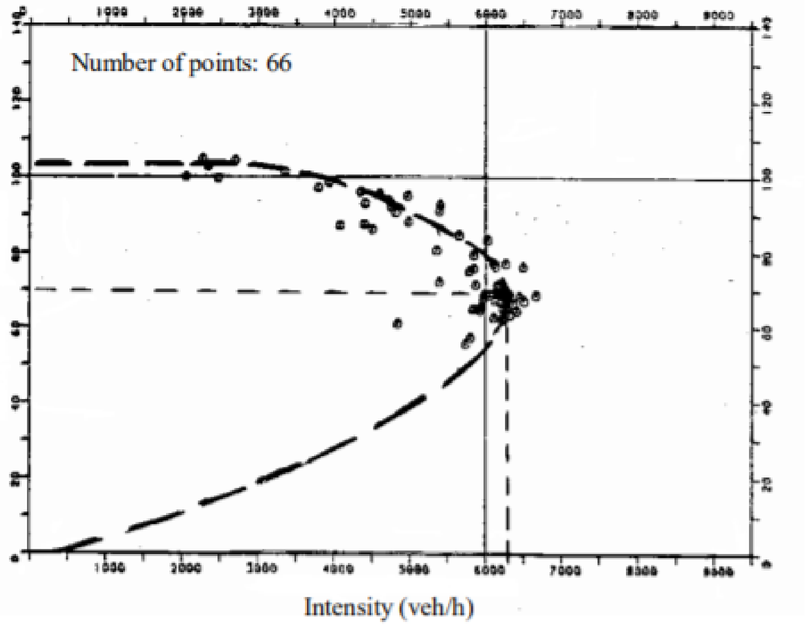
Capacity and the effect of rain

- Roadway (with a length of 4 km) with three lanes near Rotterdam under two conditions:
  - Dry and fair weather,
  - Rainy conditions.

Speed – flow in dry weather



Speed – flow in rainy weather

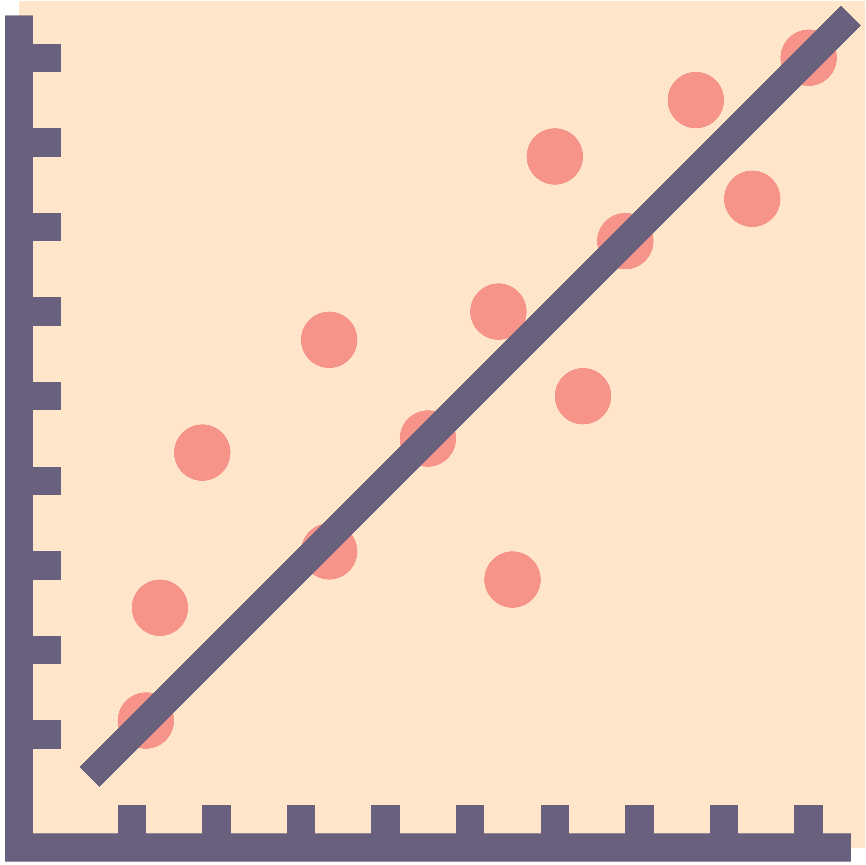


Characteristics of the capacity state

MO-team	Roadway (3 lanes)			Average per lane			
	q_c	u_c	k_c	q_c	k_c	$\bar{s}$	h
	veh/h	km/h	veh/km	veh/h	veh/km	m	s
Dry	7400	80	92.5	2460	30.8	32.4	1.46
Rain	6300	70	92.7	2100	30.9	30.9	1.71

Calibration of Macroscopic Traffic Flow Models

- The traffic models discussed thus far can be used to determine specific characteristics, such as the speed and density at which maximum flow occurs, and the jam density of a facility.
- This usually involves collecting appropriate data on the particular facility of interest and fitting a suitable model to the collected data points.
- The most common method of approach is regression analysis. This is done by minimizing the squares of the differences between the observed and expected values of a dependent variable.
- When the dependent variable is linearly related to the independent variable, the process is known as linear regression analysis.
- When the relationship is with two or more independent variables, the process is known as multiple regression analysis



Calibration of Macroscopic Traffic Flow Models

- If a dependent variable y and an independent variable x are related by an estimated regression function, then:

$$y = a + bx$$

- The constants a and b could be determined from:

$$a = \frac{1}{n}\sum_{i=1}^{n} y_i - \frac{b}{n}\sum_{i=1}^{n} x_i = \bar{y} - b\bar{x}$$

$$b = \frac{\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2}$$

  - Where:
    - n number of sets of observations
    - x_i i^{th} observation for x
    - y_i i^{th} observation for y

- A measure commonly used to determine the suitability of an estimated regression function is the coefficient of determination

$$R^2 = \frac{\sum_{i=1}^{n} (Y_i - \bar{y})^2}{\sum_{i=1}^{n} (y_i - \bar{y})^2}$$

  - Where Y_i is the value of the dependent variable as computed from the regression equations. The closer R^2 is to 1, the better the regression fits.

Calibration of Macroscopic Traffic Flow Models

Fitting Speed and Density Data to the Greenshields Model

- Let us now use the data shown in the table (columns 1 and 2) to demonstrate the use of the method of regression analysis in fitting speed and density data to the Greenshields model:

$$\bar{u}_s = u_f - \frac{u_f}{k_j} k$$

- Comparing this expression with the regression function presented before, we see that the speed $\bar{u}_s$ in the Greenshields expression is represented by y in the estimated regression function, the mean free speed u_f is represented by a, and the value of the mean free speed u_f divided by the jam density k_j is represented by b. We therefore obtain:

$$\sum y_i = 404.8 \qquad \sum x_i = 892 \qquad \bar{y} = 28.91$$

$$\sum x_i y_i = 20619.8 \qquad \sum x_i^2 = 66,628 \qquad \bar{x} = 63.71$$

Speed and Density Observations at a Rural Road

Speed, u_s (mi/h) y_i	Density, k (veh/mi) x_i	$x_i y_i$	x_i^2
53.2	20	1064.0	400
48.1	27	1298.7	729
44.8	35	1568.0	1,225
40.1	44	1764.4	1,936
37.3	52	1939.6	2,704
35.2	58	2041.6	3,364
34.1	60	2046.0	3,600
27.2	64	1740.8	4,096
20.4	70	1428.0	4,900
17.5	75	1312.5	5,625
14.6	82	1197.2	6,724
13.1	90	1179.0	8,100
11.2	100	1120.0	10,000
8.0	115	920.0	13,225
$\Sigma = 404.8$	$\Sigma = 892$	$\Sigma = 20,619.8$	$\Sigma = 66,628.0$
$\bar{y} = 28.91$	$\bar{x} = 63.71$		

Calibration of Macroscopic Traffic Flow Models

Fitting Speed and Density Data to the Greenshields Model

- We therefore obtain

$$a = 28.91 - 63.71b \quad \text{, and} \quad b = \frac{20{,}619.8 - \dfrac{(892)(4048)}{14}}{66{,}628 - \dfrac{(892)^2}{14}} = -0.53 \quad \Rightarrow \quad a = 28.91 - 63.71(-0.53) = 62.68$$

- Since $a = u_f = 62.68\ mi/h$, and $\frac{u_f}{k_j} = 0.53$ $\Rightarrow$ $k_j = 118\ veh/mi$

$\Rightarrow$ $\bar{u}_s = 62.68 - 0.53k$

- Calculating R^2 we obtain: $R^2 = 0.95$
- The maximum flow: $q_m = \frac{u_f k_j}{4} = \frac{118 \times 62.68}{4} = 1849\ veh/h$
- The velocity at which flow is maximum: $\frac{62.68}{2} = 31.3\ mi/h$
- The density at which flow is maximum: $\frac{118}{2} = 59\ veh/mi$

Calibration of Macroscopic Traffic Flow Models

Fitting Speed and Density Data to the Greenberg Model

- the Greenberg model can be written as:

$$\bar{u}_s = c \ln \frac{k_j}{k} \quad \rightarrow \quad \bar{u}_s = c \ln k_j - c \ln k$$

- $\bar{u}_s$ in the Greenberg expression is represented by y in the estimated regression function, $c \ln k_j$ is represented by a, c ($c = u_m$, the speed for maximum flow) is represented by $-b$, and $\ln k$ is represented by x.

$$\sum y_i = 404.8 \qquad \sum x_i = 56.72 \qquad \bar{y} = 28.91$$

$$\sum x_i y_i = 1547.02 \qquad \sum x_i^2 = 233.04 \qquad \bar{x} = 4.05$$

Speed and Density Observations at a Rural Road

Speed, u_s (mi/h) y_i	Density, k (veh/mi)	Ln $k_i x_i$	$x_i y_i$	x_i^2
53.2	20	2.995732	159.3730	8.974412
48.1	27	3.295837	158.5298	10.86254
44.8	35	3.555348	159.2796	12.64050
40.1	44	3.784190	151.746	14.32009
37.3	52	3.951244	147.3814	15.61233
35.2	58	4.060443	142.9276	16.48720
34.1	60	4.094344	139.6171	16.76365
27.2	64	4.158883	113.1216	17.29631
20.4	70	4.248495	86.66929	18.04971
17.5	75	4.317488	75.55605	18.64071
14.6	82	4.406719	64.33811	19.41917
13.1	90	4.499810	58.94750	20.24828
11.2	100	4.605170	51.57791	21.20759
8.0	115	4.744932	37.95946	22.51438
$\Sigma = 404.8$		$\Sigma = 56.71864$	$\Sigma = 1547.024$	$\Sigma = 233.0369$
$\bar{y} = 28.91$		$\bar{x} = 4.05$		

Calibration of Macroscopic Traffic Flow Models

Fitting Speed and Density Data to the Greenshields Model

- We therefore obtain

$$a = 28.91 - 4.05b \quad \text{, and} \quad b = \frac{1547.02 - \dfrac{(56.72)(404.8)}{14}}{233.04 - \dfrac{56.72^2}{14}} = -28.68 \quad \Rightarrow \quad a = 28.91 - 4.05(-28.68) = 145.06$$

- Since $\mathrm{a} = 145.06$, and $\mathrm{b} = -28.68 \Rightarrow u_f = 28.68\ mi/h$

- Therefore: $c \ln k_j = 145.06$

$$\ln k_j = \frac{145.06}{28.68} = 5.06$$

$$k_j = 157 \text{ veh/mi} \Rightarrow \left\{ \begin{array}{l} R^2 = 0.95 \\ \bar{u}_s = 28.68 \ln \dfrac{157}{k} \end{array} \right\}$$

- To obtain

$$\bar{u}_s k = q = ck \ln \frac{k_j}{k} \Rightarrow \frac{dq}{dk} = c \ln \frac{k_j}{k} - c \xrightarrow{\text{For maximum flow}} \frac{dq}{dk} = 0 \Rightarrow \ln \frac{k_j}{k_o} = 1$$

Obtaining k_0, the density for maximum flow: $\Rightarrow$

$$\left\{ \begin{array}{r} \ln k_j = 1 + \ln k_o \\ \ln 157 = 1 + \ln k_o \\ 5.06 = 1 + \ln k_o \\ 58.0 = k_o \\ q_{\max} = 58.0 \times 28.68 \text{ veh/h} \\ q_{\max} = 1663 \text{ veh/h} \end{array} \right.$$

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Thank you
ЯСЕНЕВО