The neutron diffusion equation

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Content (week 2)

- Full core calculations
  - The steady-state diffusion equation
  - Continuity and boundary conditions
- The finite homogeneous reactor
  - Cartesian coordinate system
  - Spherical coordinate system
  - Cylindrical coordinate system
- The heterogeneous 1D slab reactor
  - Two region example
The full-core calculation consists of solving a simplified transport equation, either the diffusion equation or the simplified $P_n$ equation.

This solution can be performed either in transient or steady-state conditions, using a small number of energy groups (generally, $G = 2$ is sufficient).

A steady-state full-core calculation generally uses the effective multiplication factor $K_{\text{eff}}$ as eigenvalue. Another possible choice of eigenvalue is to select a poison concentration or the position of a reactivity device.

Full-core calculations offer the possibility of accurately representing the reactor boundary. A correct representation of neutron leakage will be possible, as neutrons effectively escape the reactor domain through its boundary.
Full core calculations

The neutron diffusion equation – 4/31

ENE6103: Week 2

$K_{\text{eff}} B^2$

secondary fission neutrons

leakage group 1

leakage group 2

leakage group 3

leakage group $G$

neutron group 1

neutron group 2

neutron group 3

neutron group $G$

absorption group 1

absorption group 2

absorption group 3

absorption group $G$

fission group 1

fission group 2

fission group 3

fission group $G$

Energy

10 MeV

0 ev
We will first investigate the steady-state solution of the transport equation over the complete reactor domain. The neutron balance over any control domain in energy group $g$ is written

\[
\text{Leakage rate} + \text{Collision rate} = \text{Sources}
\]

or, in symbolic form,

\[
\nabla \cdot J_g(r) + \Sigma_g(r)\phi_g(r) = Q_g(r)
\]

where $J_g(r)$ is the neutronic current. The scalar product of the neutronic current with $N$ is equal to the net number of neutrons crossing the arbitrary surface per unit surface and time.
The neutronic sources $Q_g(r)$ represent the production of secondary neutrons from scattering (including neutrons from $(n,xn)$ reactions) and fission reactions:

\begin{equation}
Q_g(r) = \sum_{h=1}^{G} \Sigma_{g \leftarrow h}(r) \phi_h(r) + \frac{\chi_g(r)}{K_{\text{eff}}} \sum_{h=1}^{G} \nu \Sigma_{fh}(r) \phi_h(r)
\end{equation}

where

\begin{align*}
G &= \text{total number of energy groups} \\
\Sigma_{g \leftarrow h}(r) &= \text{macroscopic scattering cross section from group } h \text{ toward group } g \\
\chi_g(r) &= \text{fission spectrum in group } g \\
\nu \Sigma_{fh}(r) &= \text{product of the macroscopic fission cross section by the average number of neutrons emitted per fission in group } h.
\end{align*}
In the particular case where $G = 2$, two approximations will be made:

- a neutron cannot be accelerated from group 2 (thermal) toward group 1 (fast)
- all the secondary neutrons from fission are produced in group 1.

The approximations are written

$$\Sigma_{1 \leftarrow 2}(r) = 0, \quad \chi_1(r) = 1 \quad \text{and} \quad \chi_2(r) = 0.$$  \hfill (1)

Using these approximations, the sources $Q_g(r)$ simplify to

$$Q_1(r) = \Sigma_{1 \leftarrow 1}(r)\phi_1(r) + \frac{1}{K_{\text{eff}}} \left[ \nu \Sigma_{f1}(r)\phi_1(r) + \nu \Sigma_{f2}(r)\phi_2(r) \right]$$  \hfill (2)

$$Q_2(r) = \Sigma_{2 \leftarrow 1}(r)\phi_1(r) + \Sigma_{2 \leftarrow 2}(r)\phi_2(r).$$  \hfill (3)
We have obtained a balance equation for a steady-state reactor, corresponding to the situation where the leakage and absorption rates are exactly equal to the production rate of new neutrons, at all times, in each energy group.

In this case, the effective multiplication factor $K_{\text{eff}}$ is an artifact (or an eigenvalue) that satisfies this equality. This factor is expected to be close to one for a reactor in a nominal situation.

Later, we will show how an additional term can be added to the transport equation to represent transient behavior of the reactor in cases where the equality is not met.

Equation (1) must be solved on the scale of the complete reactor, correctly taking into account the position of its boundaries.

Solution of Eq. (1) requires additional information to relate the neutron flux and current. Two approaches are possible to obtain this information:

- use a spherical harmonics ($P_n$) or discrete ordinate ($S_N$) approach. A legacy alternative is to use a variant of the $P_n$ method based on the simplified $P_n$ equation.

- relate the neutron current to the gradient of the neutron flux using the Fick law. This second choice will lead to the diffusion equation.
The Fick law is a heuristic relation between the neutron current and the gradient of the neutron flux, translating the fact that neutrons have a tendency to migrate from regions where they are more numerous to regions where they are less. This relation is known to be acceptable on the scale of the complete reactor, but not at the level of lattice calculations where it breaks down. It is written

\[ J_g(r) = - \mathbb{D}_g(r) \nabla \phi_g(r) \]  

(4)

where \( \mathbb{D}_g(r) \) is a 3 \( \times \) 3 diagonal tensor containing directional diffusion coefficients.

Non-directional diffusion coefficients are generally sufficient to represent streaming effects in the lattice. In this case, the three diagonal components are simply set to the same value. Directional diffusion coefficients are more closely related to the \( B_1 \) heterogeneous streaming effect occurring when long streaming channels or planes are open in the reactor. The unit of the diffusion coefficient is the centimeter (cm).

Substituting Eq. (4) into Eq. (1), we get the neutron diffusion equation as

\[ - \nabla \cdot \mathbb{D}_g(r) \nabla \phi_g(r) + \Sigma_g(r) \phi_g(r) = Q_g(r) . \]  

(5)
The steady-state diffusion equation

The next step consists in subtracting the within-group scattering rate from both sides of Eq. (5). We obtain the one-speed neutron diffusion equation as

\[ - \nabla \cdot D_g(r) \nabla \phi_g(r) + \Sigma_{rg}(r) \phi_g(r) = Q^\diamond_g(r) \]

(6)

where \( \Sigma_{rg}(r) = \Sigma_g(r) - \Sigma_g^{-g}(r) \) is the removal cross section and where \( Q^\diamond_g(r) \) is written as

\[ Q^\diamond_g(r) = \sum_{\substack{h=1 \atop h \neq g}}^G \Sigma_{g^{-h}}(r) \phi_h(r) + \frac{\chi_g(r)}{K_{\text{eff}}} \sum_{\substack{h=1 \atop h \neq g}}^G \nu \Sigma_{fh}(r) \phi_h(r). \]

(7)

In the two–energy group case \((G = 2)\), Eq. (7) simplifies to

\[ Q^\diamond_1(r) = \frac{1}{K_{\text{eff}}} \left[ \nu \Sigma_{f1}(r) \phi_1(r) + \nu \Sigma_{f2}(r) \phi_2(r) \right] \]

\[ Q^\diamond_2(r) = \Sigma_{2^{-1}}(r) \phi_1(r). \]

(8)

The neutron diffusion equation can be solved analytically in academic cases or using standard numerical analysis techniques such as the finite difference or finite element method.
Substituting the source term from Eq. (7) into Eq. (6), we get the multigroup form of the steady-state neutron diffusion equation:

\[- \nabla \cdot \mathbb{D}_g(\mathbf{r}) \nabla \phi_g(\mathbf{r}) + \Sigma_{rg}(\mathbf{r}) \phi_g(\mathbf{r})\]

\[= \sum_{g=1}^{G} \Sigma_{g-h}(\mathbf{r}) \phi_h(\mathbf{r}) + \frac{\chi_g(\mathbf{r})}{\lambda} \sum_{h=1}^{G} \nu \Sigma_{fh}(\mathbf{r}) \phi_h(\mathbf{r}) .\]

Equation (9) is an eigenproblem, whose solution behaves in a typical way:

- A trivial solution of Eq. (9) is \( \phi_g(\mathbf{r}) = 0 \), \( \forall \mathbf{r} \). Many non-trivial solutions of Eq. (9) exist for different eigenvalues \( \lambda \). The largest eigenvalue in absolute value corresponds to the fundamental solution of the eigenproblem and is equal to the effective multiplication factor \( K_{\text{eff}} \) of the reactor. Only the fundamental solution has a physical meaning. The eigenspectrum of Eq. (9) is the set of all its eigenvalues (including \( K_{\text{eff}} \)).

- Only the fundamental solution can lead to a positive neutron flux \( \phi_g(\mathbf{r}) \) over the reactor domain. The other solutions are called neutron flux harmonics and lead to oscillating values of the flux, sometime positive, sometime negative.
The steady-state diffusion equation

Every solution of Eq. (9) can be renormalized with an arbitrary normalization constant. If $\phi_g(r)$ is a solution, then $C \phi_g(r)$ is also a solution for any value of constant $C$. The normalization constant is generally computed from the knowledge of the reactor power:

$$
\sum_{g=1}^{G} \int_V d^3r \, H_g(r) \, \phi_g(r) = P
$$

(10)

where $V$ is the volume of the reactor, $H_g(r)$ is the H–factor and $P$ is the power of the reactor. The H–factor permits the computation of the recoverable energy produced by the reactor.
It is possible to find a mathematical adjoint to Eq. (9). This adjoint equation has the same eigenspectrum as Eq. (9).

The mathematical adjoint of Eq. (9) is obtained by permuting primary and secondary group indices. It is written

\[- \nabla \cdot D_g(r) \nabla \phi^*_g(r) + \sum_{r} g(r) \phi^*_g(r)\]

(11)

\[= \sum_{h=1}^{G} \Sigma_{h \leftarrow g} (r) \phi^*_h (r) + \frac{\nu \Sigma_{f} g(r)}{\lambda} \sum_{h=1}^{G} \chi_h (r) \phi^*_h (r)\]

where $\phi^*_g(r)$ is the adjoint flux or adjoint flux harmonics. The particular case with $G = 1$ is said to be self-adjoint as it corresponds to the case where $\phi^*_g(r) = \phi_g(r)$. The adjoint flux can also be renormalized with an arbitrary normalization constant. It is generally normalized using the following arbitrary relation:

\[\sum_{g=1}^{G} \int_V d^3 r \ X_f g(r) \phi^*_g(r) = 1.\]
Continuity and boundary conditions

The neutron flux is a continuous distribution of $r$ and the neutron current must be continuous across an infinite plane placed at abscissa $x_0$. The flux continuity condition at this point is

$$\phi_g(x_0^-, y, z) = \phi_g(x_0^+, y, z) \quad \forall \ y \text{ and } z.$$  \hfill (13)

The neutron current continuity condition is written after introducing the unit normal $N = (1, 0, 0)$, perpendicular to the infinite plane. We write

$$J_g(x_0^-, y, z) \cdot N = J_g(x_0^+, y, z) \cdot N \quad \forall \ y \text{ and } z.$$  \hfill (14)

Using the Fick law from Eq. (4), we obtain

$$- D_g(x_0^-, y, z) \nabla \phi_g(x_0^-, y, z) \cdot N = - D_g(x_0^+, y, z) \nabla \phi_g(x_0^+, y, z) \cdot N \quad \forall \ y \text{ and } z$$  \hfill (15)

or

$$D_g(x_0^-, y, z) \frac{d}{dx} \phi_g(x, y, z) \bigg|_{x=x_0^-} = D_g(x_0^+, y, z) \frac{d}{dx} \phi_g(x, y, z) \bigg|_{x=x_0^+} \quad \forall \ y \text{ et } z.$$  \hfill (16)

Equation (16) indicates that the neutron flux gradient is discontinuous at each point of the domain where the diffusion coefficient is discontinuous.
Continuity and boundary conditions

Boundary conditions must be applied at each point of \( \partial V \).

The most straightforward way to represent a real boundary is to set a zero-flux boundary condition on it. In this case, we simply write

\[
\phi_g(\mathbf{r}) = 0 \quad \text{if} \quad \mathbf{r} \in \partial W_i
\]

(17)

This condition is not satisfactory, as it assumes the absence of neutrons on \( \partial W_i \).

It would be more exact to assume a zero incoming current condition. This condition is a particular case of the more general albedo boundary condition. We first identify the

incoming and outgoing net currents, using

\[
\mathbf{J}_g(\mathbf{r}) \cdot \mathbf{N}(\mathbf{r}) = J^+_g(\mathbf{r}) - J^-_g(\mathbf{r}) \quad \text{if} \quad \mathbf{r} \in \partial W_i
\]

where \( \mathbf{N}(\mathbf{r}) \) is a unit vector, normal to \( \partial W_i \), and pointing in the outgoing direction.

The albedo at \( \mathbf{r} \) is defined by the relation

\[
\beta_g(\mathbf{r}) = \frac{J^-_g(\mathbf{r})}{J^+_g(\mathbf{r})} \quad \text{if} \quad \mathbf{r} \in \partial W_i
\]

(18)

In general, we use the same albedo in all energy groups. The most usual values are \( \beta(\mathbf{r}) = 0 \) to represent a zero incoming current condition and \( \beta(\mathbf{r}) = 1 \) to represent a symmetry condition.
Continuity and boundary conditions

The incoming and outgoing net currents can be obtained from the neutron current and flux distributions provided the angular flux is represented by a limited $P_1$ expansion

\begin{align*}
J_g^-(r) &= \frac{1}{4} \phi_g(r) - \frac{1}{2} J_g(r) \cdot N(r) \\
J_g^+(r) &= \frac{1}{4} \phi_g(r) + \frac{1}{2} J_g(r) \cdot N(r).
\end{align*}

Substituting Eqs. (19) into (18) and using the Fick law (4), we obtain the albedo boundary condition as

\begin{equation}
\mathbb{D}_g(r) \nabla \phi_g(r) \cdot N(r) + \frac{1}{2} \frac{1 - \beta(r)}{1 + \beta(r)} \phi_g(r) = 0 \quad \text{if } r \in \partial W_i
\end{equation}

where $\partial W_i$ is the fraction of $\partial V$ where the albedo boundary condition is applied.

The Fick law and the $P_1$ approximation introduce an error in the zero incoming current condition that can be reduced by using a value of $\beta(r)$ slightly above zero. We recommend to represent the zero incoming current condition using the value $\beta(r) = 0.031758$.

Another particular case is the symmetry condition obtained by setting the albedo to value $\beta(r) = 1$. In this case, Eq. (20) reduces to

\begin{equation}
\nabla \phi_g(r) \cdot N(r) = 0 \quad \text{if } r \in \partial W_i.
\end{equation}
The finite homogeneous reactor

Consider an homogeneous and finite reactor surrounded by zero-flux or symmetry boundary conditions. The nuclear properties of the reactor are independent of space.

We use non-directional diffusion coefficients.

Equation (9) simplifies to

\[
-D_g \nabla^2 \phi_g(r) + \Sigma_{rg} \phi_g(r) = \sum_{h=1}^{G} \Sigma_{g \leftarrow h} \phi_h(r) + \frac{\chi_g}{K_{\text{eff}}} \sum_{h=1}^{G} \nu \Sigma_{fh} \phi_h(r).
\]

Equation (22)

It is possible to factorize the flux according to

\[
\phi_g(r) = \psi(r) \varphi_g.
\]

Equation (23)

Substituting Eq. (23) into Eq. (22), we obtain

\[
-\frac{\nabla^2 \psi(r)}{\psi(r)} = -\frac{\Sigma_{rg}}{D_g} + \frac{1}{D_g \varphi_g} \left\{ \sum_{h=1}^{G} \Sigma_{g \leftarrow h} \varphi_h + \frac{\chi_g}{K_{\text{eff}}} \sum_{h=1}^{G} \nu \Sigma_{fh} \varphi_h \right\}.
\]

Equation (24)
We note that the left side of Eq. (24) is independent of the neutron energy whereas its right side is independent of the position in reactor. This fact is only possible if each side of Eq. (24) is itself equal to the same constant. This constant was set equal to $B^2$, the buckling of the reactor. We therefore obtain two independent equations as

\[ \nabla^2 \psi(r) + B^2 \psi(r) = 0 \]  

(25)

and

\[ \left[ D_g B^2 + \Sigma_{rg} \right] \varphi_g = \sum_{h=1}^{G} \Sigma_{g \leftarrow h} \varphi_h + \frac{X_g}{K_{eff}} \sum_{h=1}^{G} \nu \Sigma_{fh} \varphi_h. \]  

(26)

Equation (25) is a Laplace equation, an eigenproblem whose eigenvalue is the buckling $B^2$. Its solution is a function of the shape and size of the reactor and of the boundary conditions:

- **Zero-flux boundary condition:** $\psi(r) = 0$ if $r \in \partial W_i$
- **Symmetry boundary condition:** $\nabla \psi(r) \cdot N(r) = 0$ if $r \in \partial W_i$. 

Equation (25) has many non-trivial solutions, each of them corresponding to an element of its eigenspectrum, but only the fundamental solution corresponds to a positive neutron flux everywhere in the domain.
Cartesian coordinate system

The Cartesian coordinate system is the most usual choice for real application problems. In this case, the Laplace operator is written

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} .$$

(27)

Let us consider a prismatic homogeneous reactor of dimension $L_x \times L_y \times L_z$. In this case, the fundamental solution of Eq. (25) is

$$\psi(x, y, z) = C \sin \frac{\pi x}{L_x} \sin \frac{\pi y}{L_y} \sin \frac{\pi z}{L_z} .$$

(28)

The Cartesian domain is defined over $0 \leq x \leq L_x$, $0 \leq y \leq L_y$ and $0 < z < L_z$. A zero-flux boundary condition is imposed on the surface of the domain. The normalization constant $C$ is arbitrary as both Eqs. (25) and (26) are eigenproblems.

The corresponding critical buckling is

$$B^2 = \left( \frac{\pi}{L_x} \right)^2 + \left( \frac{\pi}{L_y} \right)^2 + \left( \frac{\pi}{L_z} \right)^2 .$$

(29)
The Laplace operator is written

\[ \nabla^2 \psi = \frac{1}{r^2 \sin \theta} \left[ \sin \theta \left( r^2 \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 \psi}{\partial \epsilon^2} \right]. \]  

(30)

The fundamental solution of Eq. (25) represents the neutron flux in a spherical reactor of radius \( R \). A zero-flux boundary condition is imposed at \( r = R \) (i.e., \( \psi(R) = 0 \)). The fundamental solution is written

\[ \psi(r) = \frac{C}{r} \sin \frac{\pi r}{R} \]  

(31)

with the critical buckling equal to

\[ B^2 = \left( \frac{\pi}{R} \right)^2. \]  

(32)
The Laplace operator is written

\[
\nabla^2 \psi = \frac{1}{\rho} \left[ \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho} \frac{\partial^2 \psi}{\partial \varepsilon^2} + \rho \frac{\partial^2 \psi}{\partial z^2} \right].
\]

(33)

The fundamental solution of Eq. (25) represents the neutron flux in a cylindrical reactor of radius \( R \) and height \( L_z \). A zero-flux boundary condition is imposed on the surface. The fundamental solution is

\[
\psi(\rho, z) = C J_0 \left( \frac{2.405 \rho}{R} \right) \sin \frac{\pi z}{L_z}
\]

(34)

with the critical buckling equal to

\[
B^2 = \left( \frac{2.405}{R} \right)^2 + \left( \frac{\pi}{L_z} \right)^2
\]

(35)

where \( J_0(x) \) is a zeroth order ordinary Bessel function, such as \( J_0(2.405) = 0 \).
The heterogeneous 1D slab reactor

- The one-dimensional (1D) heterogeneous reactor configurations correspond to the case where the neutron flux is a function of a unique spatial variable.
- These cases can be solved analytically whatever the type of conditions imposed at boundaries.
- The nuclear properties of the reactor are only a function of the independent variable $x$.

Equation (9) simplifies to

$$
- \frac{d}{dx} D_g(x) \frac{d\phi_g}{dx} + \Sigma_g(x) \phi_g(x) = Q^\circ_g(x)
$$

$$
= \sum_{h=1}^{G} \Sigma_{g \leftarrow h}(x) \phi_h(x) + \frac{\chi_g(x)}{K_{\text{eff}}} \sum_{h=1}^{G} \nu \Sigma_{f h}(x) \phi_h(x).
$$

(36)

The boundary conditions are either a zero-flux condition ($\phi_g(x) = 0$) or an albedo condition:

$$
\pm D_g(x) \frac{d\phi_g}{dx} + \frac{1}{2} \frac{1 - \beta(x)}{1 + \beta(x)} \phi_g(x) = 0
$$

(37)

where the sign “−” or “+” is used for a left ($x = x_{1/2}$) or a right boundary ($x = x_{I+1/2}$).
The heterogeneous 1D slab reactor

Each slab is assumed to be homogeneous, so that the corresponding nuclear properties $D_g(x)$, $\Sigma_r g(x)$, $\Sigma_g \rightarrow h(x)$, $\chi_g(x)$ and $\nu \Sigma_{fh}(x)$ are piecewise continuous. The reactor domain is divided into $I$ regions of indices $1 \leq i \leq I$, in such a way that the nuclear properties in region $i$ are constant and equal to $D_{g,i}$, $\Sigma_{rg,i}$, $\Sigma_{g \rightarrow h,i}$, $\chi_{g,i}$ and $\nu \Sigma_{fh,i}$.

\[ - D_{g,i} \frac{d^2 \phi_g(x)}{dx^2} + \Sigma_{rg,i} \phi_g(x) = Q^g(x) = \sum_{h=1}^{G} \Sigma_{g \rightarrow h,i} \phi_h(x) + \frac{\chi_{g,i}}{K_{\text{eff}}} \sum_{h=1}^{G} \nu \Sigma_{fh,i} \phi_h(x) \]

if $x_{i-1/2} < x < x_{i+1/2}$.

\[ (38) \]
At this point, we introduce the analytical solution approach for Eq. (38). It is based on a linear transformation technique, valid only for multigroup 1D problems. Equation (38) is first rewritten in matrix form as

\[ \frac{d^2}{dx^2} \Phi(x) + F_i \Phi(x) = 0 \quad \text{if } x_{i-1/2} < x < x_{i+1/2} \]

with

\[ \Phi(x) = \begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_G(x) \end{pmatrix} \quad \text{and} \quad F_i = \begin{pmatrix} f_{11,i} & f_{12,i} & \cdots & f_{1G,i} \\ f_{21,i} & f_{22,i} & \cdots & f_{2G,i} \\ \vdots & \vdots & \ddots & \vdots \\ f_{G1,i} & f_{G2,i} & \cdots & f_{GG,i} \end{pmatrix} \]

where the components \( f_{gh,i} \) of this matrix are written as

\[ f_{gh,i} = \frac{1}{D_{g,i}} \left[ -\Sigma_{r_g,i} \delta_{gh} + \Sigma_{g<\rightarrow h,i} (1 - \delta_{gh}) + \frac{\chi_{g,i}}{K_{\text{eff}}} \nu \Sigma_{f,h,i} \right]. \]
The heterogeneous 1D slab reactor

The next step consists in finding all eigenvectors $t_{\ell,i}$ of matrix $F_i$ with the associated eigenvalues $\lambda_{\ell,i}$. We build a matrix $T_i$ whose columns are the eigenvectors of $F_i$:

$$T_i = \begin{pmatrix} t_{1,i} & t_{2,i} & \cdots & t_{G,i} \end{pmatrix}$$

so that

$$F_i T_i = T_i \text{diag}(\lambda_{\ell,i}) .$$

The linear transformation technique used to solve Eq. (36) is based on the introduction of an unknown vector $\Psi(x)$ defined in such a way that

$$\Phi(x) = T_i \Psi(x) = \begin{pmatrix} t_{11,i} & t_{12,i} & \cdots & t_{1G,i} \\ t_{21,i} & t_{22,i} & \cdots & t_{2G,i} \\ \vdots & \vdots & \ddots & \vdots \\ t_{G1,i} & t_{G2,i} & \cdots & t_{GG,i} \end{pmatrix} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_G(x) \end{pmatrix}$$

and to its substitution in Eq. (39). We obtain

$$\frac{d^2}{dx^2} T_i \Psi(x) + F_i T_i \Psi(x) = 0 \quad \text{if } x_{i-1/2} < x < x_{i+1/2} .$$
We next left-multiply each side of Eq. (45) by $[T_i]^{-1}$ and use Eq. (43) to obtain

\begin{equation}
\frac{d^2}{dx^2} \Psi(x) + \text{diag}(\lambda_{\ell,i}) \Psi(x) = 0 \quad \text{if } x_{i-1/2} < x < x_{i+1/2}.
\end{equation}

Equation (46) is similar to Eq. (39) with the difference that all the energy groups are uncoupled. Its resolution is reduced to the solution of $G$ one-speed problems. In each energy group $g$, we assume an analytical solution of the form

\begin{equation}
\psi_g(x) = \begin{cases} 
A_{g,i} \cos(\sqrt{\lambda_{g,i}} x) + B_{g,i} \sin(\sqrt{\lambda_{g,i}} x) & \text{if } \lambda_{g,i} \geq 0; \\
C_{g,i} \cosh(\sqrt{-\lambda_{g,i}} x) + E_{g,i} \sinh(\sqrt{-\lambda_{g,i}} x) & \text{otherwise}
\end{cases}
\end{equation}

if $x_{i-1/2} < x < x_{i+1/2}$.

The analytical expression of the flux $\phi_g(x)$ is obtained after substitution of Eq. (47) into Eq. (44) as

\begin{equation}
\phi_g(x) = \sum_{h=1}^{G} t_{gh,i} \psi_h(x) \quad \text{if } x_{i-1/2} < x < x_{i+1/2}.
\end{equation}
The final step consists of coupling together the analytical solutions of each region $i$ and applying the boundary conditions. If region $i - 1$ exists (i.e., if the left side of region $i$ is not a boundary), the solution obeys the continuity relations (13) and (16), so that

$$\phi_g(x_{i-1/2}^-) = \phi_g(x_{i-1/2}^+)$$

(49)

and

$$D_{g,i-1} \phi'_g(x_{i-1/2}^-) = D_{g,i} \phi'_g(x_{i-1/2}^+)$$

(50)

where we defined $\phi'_g(x_{i-1/2}^-) = \frac{d\phi_g}{dx} \bigg|_{x = x_{i-1/2}}$. Similarly, if region $i + 1$ exists, we write

$$\phi_g(x_{i+1/2}^-) = \phi_g(x_{i+1/2}^+)$$

(51)

and

$$D_{g,i} \phi'_g(x_{i+1/2}^-) = D_{g,i+1} \phi'_g(x_{i+1/2}^+)$$

(52)

The boundary conditions are imposed in the same way, by forcing the value of $\phi_g(x)$ or $\phi'_g(x)$ on a boundary abscissa.

An analytical solution of Eq. (38) with its continuity and boundary conditions can always be found, with a computational effort increasing with the number of energy groups and regions.
We apply this technique to a one-speed, two-region problem. Eq. (38) simplifies to

\[- D_i \frac{d^2 \phi}{dx^2} + \Sigma_{r,i} \phi(x) = \frac{1}{K_{\text{eff}}} \nu \Sigma_{f,i} \phi(x) \]

if \( x_{i-1/2} < x < x_{i+1/2} \).

We assume an analytical solution of the form

\[
\phi(x) = \begin{cases} 
A_{11} \cos(\kappa_1 x) + A_{21} \sin(\kappa_1 x) & \text{if } 0 \leq x \leq \frac{1}{2}; \\
A_{12} \cos(\kappa_2 x) + A_{22} \sin(\kappa_2 x) & \text{if } \frac{1}{2} \leq x \leq 1
\end{cases}
\]

\[
\phi'(x) = \begin{cases} 
-A_{11} \kappa_1 \sin(\kappa_1 x) + A_{21} \kappa_1 \cos(\kappa_1 x) & \text{if } 0 \leq x \leq \frac{1}{2}; \\
-A_{12} \kappa_2 \sin(\kappa_2 x) + A_{22} \kappa_2 \cos(\kappa_2 x) & \text{if } \frac{1}{2} \leq x \leq 1
\end{cases}
\]

\[
\frac{d^2 \phi}{dx^2} = \begin{cases} 
-A_{11} \kappa_1^2 \cos(\kappa_1 x) - A_{21} \kappa_1^2 \sin(\kappa_1 x) & \text{if } 0 \leq x \leq \frac{1}{2}; \\
-A_{12} \kappa_2^2 \cos(\kappa_2 x) - A_{22} \kappa_2^2 \sin(\kappa_2 x) & \text{if } \frac{1}{2} \leq x \leq 1
\end{cases}
\]
The left zero-flux boundary condition is written

\[ \phi(0) = A_{11} = 0 \]  \hspace{1cm} (57)

The continuity conditions at \( x = \frac{1}{2} \) are written

\[ A_{11} \cos \left( \frac{\kappa_1}{2} \right) + A_{21} \sin \left( \frac{\kappa_1}{2} \right) = A_{12} \cos \left( \frac{\kappa_2}{2} \right) + A_{22} \sin \left( \frac{\kappa_2}{2} \right) \]  \hspace{1cm} (58)

and

\[ -A_{11} \kappa_1 D_1 \sin \left( \frac{\kappa_1}{2} \right) + A_{21} \kappa_1 D_1 \cos \left( \frac{\kappa_1}{2} \right) = -A_{12} \kappa_2 D_2 \sin \left( \frac{\kappa_2}{2} \right) + A_{22} \kappa_2 D_2 \cos \left( \frac{\kappa_2}{2} \right) \]  \hspace{1cm} (59)

Finally, the right symmetry boundary condition is written

\[ -A_{12} \kappa_2 \sin(\kappa_2) + A_{22} \kappa_2 \cos(\kappa_2) = 0 \]

so that

\[ A_{22} = A_{12} \tan(\kappa_2) \]  \hspace{1cm} (60)
Substituting Eqs. (57) and (60) into Eqs. (58) and (59), we find the following matrix equation:

\[
\begin{pmatrix}
\sin \left( \frac{\kappa_1}{2} \right) & -\frac{1}{\cos(\kappa_2)} \cos \left( \frac{\kappa_2}{2} \right) \\
\kappa_1 D_1 \cos \left( \frac{\kappa_1}{2} \right) & -\frac{\kappa_2 D_2}{\cos(\kappa_2)} \sin \left( \frac{\kappa_2}{2} \right)
\end{pmatrix}
\begin{pmatrix}
A_{21} \\
A_{12}
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

Equation (61) has a non-trivial solution only if its determinant is zero. We write

\[
\det
\begin{pmatrix}
\sin \left( \frac{\kappa_1}{2} \right) & -\frac{1}{\cos(\kappa_2)} \cos \left( \frac{\kappa_2}{2} \right) \\
\kappa_1 D_1 \cos \left( \frac{\kappa_1}{2} \right) & -\frac{\kappa_2 D_2}{\cos(\kappa_2)} \sin \left( \frac{\kappa_2}{2} \right)
\end{pmatrix}
= 0
\]

so that the resulting characteristic equation is a non-linear relation in \( \kappa_1 \) and \( \kappa_2 \), written

\[
\tan \left( \frac{\kappa_1}{2} \right) \tan \left( \frac{\kappa_2}{2} \right) = \frac{\kappa_1 D_1}{\kappa_2 D_2}.
\]

Substituting Eqs. (54) and (56) into Eq. (53), we find the expressions of \( \kappa_1 \) and \( \kappa_2 \) as

\[
\kappa_1 = \sqrt{\frac{\nu \Sigma_{f,1} - K_{\text{eff}} \Sigma_{r,1}}{K_{\text{eff}} D_1}} \quad \text{and} \quad \kappa_2 = \sqrt{\frac{\nu \Sigma_{f,2} - K_{\text{eff}} \Sigma_{r,2}}{K_{\text{eff}} D_2}}.
\]
Combining Eqs. (63) and (64), we obtain the non-linear expression of the effective multiplication factor $K_{\text{eff}}$ as

\[
\tan \left( \frac{1}{2} \sqrt{\frac{\nu \Sigma_{f,1} - K_{\text{eff}} \Sigma_{r,1}}{K_{\text{eff}} D_1}} \right) \tan \left( \frac{1}{2} \sqrt{\frac{\nu \Sigma_{f,2} - K_{\text{eff}} \Sigma_{r,2}}{K_{\text{eff}} D_2}} \right) = \frac{D_1 (\nu \Sigma_{f,1} - K_{\text{eff}} \Sigma_{r,1})}{D_2 (\nu \Sigma_{f,2} - K_{\text{eff}} \Sigma_{r,2})}.
\]

(65)

The effective multiplication factor is obtained by solving Eq. (65) using a numerical technique for the solution of a non-linear equation with one unknown. Knowledge of $K_{\text{eff}}$ permits the computation of $\kappa_1$ and $\kappa_2$, and the determination of the analytical expression of the neutron flux $\phi(x)$, the eigenvector corresponding to $K_{\text{eff}}$. Its value is

\[
\phi(x) = \begin{cases} 
A_{21} \sin(\kappa_1 x) & \text{if } 0 \leq x \leq \frac{1}{2}; \\
\frac{A_{21} \sin \left( \frac{\kappa_1}{2} \right)}{\cos \left( \frac{\kappa_2}{2} \right) \cos[\kappa_2 (1 - x)]} & \text{if } \frac{1}{2} \leq x \leq 1.
\end{cases}
\]

(66)

The neutron flux can be renormalized to the power of the reactor by adjusting the remaining constant $A_{21}$. 

The neutron diffusion equation – 31/31