The Lagrangian finite-element method

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A primal variational formulation 1

It is possible to rewrite the diffusion equation

$$-\boldsymbol{\nabla}\cdot\mathbb{D}(\boldsymbol{r})\boldsymbol{\nabla}\phi(\boldsymbol{r})+\Sigma_{\mathrm{r}}(\boldsymbol{r})\phi(\boldsymbol{r})=Q^{\diamond}(\boldsymbol{r}) \tag{1}$$

together with its continuity and boundary conditions, into a variational formulation that is mathematically equivalent.

- This approach is also known as the Rayleigh-Ritz method and provides a means to obtain approximate solutions to a differential equation by finding a stationary point of a related functional.
- Many variational formulations exist, but we will restrict ourself to the one-speed primal formulation based on the one-speed primal functional.
- This formulation enables us to obtain a solution for Eq. (1) inside a space of basis functions. The Sobolev space is the most general space that can be used.

We have suppressed the group index g in order to simplify the notation.

The Sobolev space

The Sobolev space is a vector space of functions $f(\boldsymbol{r})$ defined over domain V and bounded in the Sobolev sense. Elements of the Sobolev space $W_1(V)$ are L^2–integrable over the domain V and possess first derivatives that are also L^2–integrable over the domain V. Functions that are element of $W_1(V)$ are therefore continuous over V. We write

(2) $$W_1(V) = \left\{ f(\boldsymbol{r})\,;\ f(\boldsymbol{r}) \in L^2(V) \text{ and } \boldsymbol{\nabla} f(\boldsymbol{r}) \in \left[L^2(V)\right]^3 \right\}$$

where $L^2(V)$ is the set of functions defined over domain V, whose quadratic norm is bounded, so that

(3) $$L^2(V) = \left\{ f(\boldsymbol{r})\,;\ \sqrt{\int_V d^3r\,[f(\boldsymbol{r})]^2} < \infty \right\} \quad .$$

- The elements of $W_1(V)$ are functions with less restrictive continuity requirements than those imposed on the solution of Eq. (1).
- For example, it is possible to choose basis functions that do not satisfy the neutron current continuity relation. In this sense, a variational formulation is a weak formulation.

A primal formulation

It is also possible to chose basis functions within a subset of $W_1(V)$.

- For example, a primal formulation restrict the choice to the set $\mathcal{D}(V)$ of functions that vanish over ∂V_0, the domain boundary where a zero-flux condition is imposed.
- It is also possible to restrict ourself to piecewise polynomial functions, leading to the Lagrangian finite-element method.

A variational formulation makes it possible to solve Eq. (1) by seeking a stationary point of a corresponding functional. The one-speed primal functional is written

$$\begin{aligned} \mathcal{F}\{\phi(\boldsymbol{r})\} &= \frac{1}{2}\int_V d^3r \Big\{ D(\boldsymbol{r})\boldsymbol{\nabla}\phi(\boldsymbol{r})\cdot\boldsymbol{\nabla}\phi(\boldsymbol{r}) + \Sigma_{\mathrm{r}}(\boldsymbol{r})\,[\phi(\boldsymbol{r})]^2 \\ &- 2\phi(\boldsymbol{r})\,Q^{\diamond}(\boldsymbol{r})\Big\} + \frac{1}{2}\int_{\partial V_\beta} d^2r\,\frac{1}{2}\frac{1-\beta(\boldsymbol{r})}{1+\beta(\boldsymbol{r})}\,[\phi(\boldsymbol{r})]^2 \end{aligned} \tag{4}$$

where $\phi(\boldsymbol{r}) \in W_1(V) \cap \mathcal{D}(V)$, $\beta(\boldsymbol{r})$ is the albedo and ∂V_β is the fraction of ∂V where the albedo boundary condition is applied, so that $\partial V = \partial V_0 \cup \partial V_\beta$.

A primal formulation

A stationary point of functional (4) is defined by the relation

(5) $$\delta_{\delta\phi}\mathcal{F}\left\{\phi(\boldsymbol{r})\right\} = \lim_{\epsilon\longrightarrow 0}\left\{\frac{d}{d\epsilon}\,\mathcal{F}\left\{\phi(\boldsymbol{r}) + \epsilon\,\delta\phi(\boldsymbol{r})\right\}\right\} = 0$$

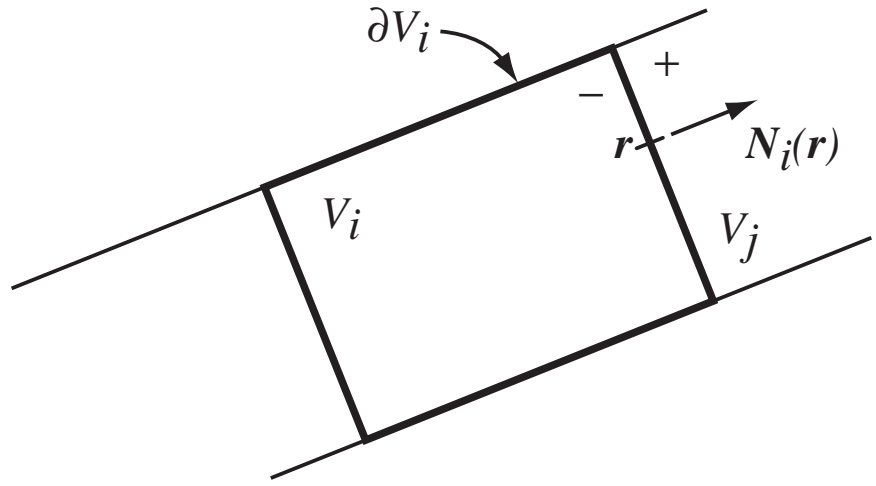
where $\delta\phi(\boldsymbol{r})$ is an arbitrary element of vector space $W_1(V) \cap \mathcal{D}(V)$. We thus obtain

(6) $$\int_V d^3r\Big\{D(\boldsymbol{r})\,\boldsymbol{\nabla}\delta\phi(\boldsymbol{r})\cdot\boldsymbol{\nabla}\phi(\boldsymbol{r}) + \Sigma_{\rm r}(\boldsymbol{r})\,\delta\phi(\boldsymbol{r})\,\phi(\boldsymbol{r}) - \delta\phi(\boldsymbol{r})\,Q^{\diamond}(\boldsymbol{r})\Big\} + \int_{\partial V_\beta} d^2r\,\frac{1}{2}\frac{1-\beta(\boldsymbol{r})}{1+\beta(\boldsymbol{r})}\,\delta\phi(\boldsymbol{r})\,\phi(\boldsymbol{r}) = 0$$

$\forall\delta\phi(\boldsymbol{r}) \in W_1(V) \cap \mathcal{D}(V)$.

A primal formulation

We divide the total reactor volume V into a set of subvolumes $\{V_i\ ;\ i=1,N\}$ over which constant diffusion coefficient $\{D_i\ ;\ i=1,N\}$ and constant removal cross sections $\{\Sigma_{\mathrm{r},i}\ ;\ i=1,N\}$ are defined. Moreover, $\boldsymbol{\nabla}\phi(\boldsymbol{r})$ and $\boldsymbol{\nabla}\delta\phi(\boldsymbol{r})$ are assumed continuous inside each subvolume V_i. We also define $\{\partial V_i\ ;\ i=1,N\}$ as surfaces surrounding each subvolume and $\boldsymbol{N}_i(\boldsymbol{r})$ as the normal unit vector pointing out of ∂V_i at $\boldsymbol{r}$. We note that $\boldsymbol{N}_i(\boldsymbol{r}) = -\boldsymbol{N}_j(\boldsymbol{r})$.



Let us now apply the Gauss divergence theorem over subvolume V_i, so that

$$\int_{V_i} d^3r\, D(\boldsymbol{r})\boldsymbol{\nabla}\delta\phi(\boldsymbol{r})\cdot\boldsymbol{\nabla}\phi(\boldsymbol{r})$$

$$= -\int_{V_i} d^3r\, \delta\phi(\boldsymbol{r})\nabla\cdot D(\boldsymbol{r})\boldsymbol{\nabla}\phi(\boldsymbol{r}) + \int_{\partial V_i} d^2r\, \delta\phi(\boldsymbol{r})\, D(\boldsymbol{r}_-)\,\boldsymbol{\nabla}\phi(\boldsymbol{r}_-)\cdot\boldsymbol{N}_i(\boldsymbol{r}) \tag{7}$$

where $i=1,N$.

A primal formulation

We separate each surface ∂V_i into three parts, as

(8) $$\partial V_i = \partial W_i + \partial V_{\beta,i} + \partial V_{0,i}$$

where ∂W_i is the part of ∂V_i internal to domain V. Surfaces $\partial V_{\beta,i}$ and $\partial V_{0,i}$ are the part of ∂V_i belonging to ∂V_β an ∂V_0, respectively.

Substituting Eq. (7) into Eq. (6), we obtain

(9) $$\begin{aligned}\sum_{i=1}^{N} \int_{V_i} d^3r\, \delta\phi(\boldsymbol{r}) \Big[-\nabla\cdot D(\boldsymbol{r})\boldsymbol{\nabla}\phi(\boldsymbol{r}) + \Sigma_{\mathrm{r}}(\boldsymbol{r})\,\phi(\boldsymbol{r}) - Q^{\diamond}(\boldsymbol{r})\Big] \\ + \int_{\partial W_i} d^2r\, \delta\phi(\boldsymbol{r})\, D(\boldsymbol{r}_-)\, \boldsymbol{\nabla}\phi(\boldsymbol{r}_-)\cdot \boldsymbol{N}_i(\boldsymbol{r}) \\ + \int_{\partial V_{\beta,i}} d^2r\, \delta\phi(\boldsymbol{r}) \left[D(\boldsymbol{r})\, \boldsymbol{\nabla}\phi(\boldsymbol{r})\cdot \boldsymbol{N}_i(\boldsymbol{r}) + \frac{1}{2}\frac{1-\beta(\boldsymbol{r})}{1+\beta(\boldsymbol{r})}\,\phi(\boldsymbol{r})\right] = 0\end{aligned}$$

$\forall \delta\phi(\boldsymbol{r}) \in W_1(V) \cap \mathcal{D}(V)$.

A primal formulation

Hence $\phi(\boldsymbol{r})$ is a stationnary point of functional (4) for all arbitrary variations $\delta\phi(\boldsymbol{r}) \in W_1(V) \cap \mathcal{D}(V)$ if, and only if its Euler equations are satisfied:

(10) $$-\nabla \cdot D(\boldsymbol{r})\boldsymbol{\nabla}\phi(\boldsymbol{r}) + \Sigma_{\mathrm{r}}(\boldsymbol{r})\,\phi(\boldsymbol{r}) = Q^{\diamond}(\boldsymbol{r})\,; \quad \boldsymbol{r} \in V \quad ,$$

(11) $$\sum_{i=1}^{N} \delta_i(\boldsymbol{r})\, D_i\, \boldsymbol{\nabla}\phi(\boldsymbol{r}_-) \cdot \boldsymbol{N}_i(\boldsymbol{r}) = 0 \quad \text{with} \quad \delta_i(\boldsymbol{r}) = \begin{cases} 1 & \text{if } \boldsymbol{r} \in \delta W_i \\ 0 & \text{otherwise} \end{cases}$$

and

(12) $$D(\boldsymbol{r})\,\boldsymbol{\nabla}\phi(\boldsymbol{r}) \cdot \boldsymbol{N}(\boldsymbol{r}) + \frac{1}{2}\frac{1-\beta(\boldsymbol{r})}{1+\beta(\boldsymbol{r})}\,\phi(\boldsymbol{r}) = 0\,; \quad \boldsymbol{r} \in \partial V_\beta \quad .$$

In conclusion, a function $\phi(\boldsymbol{r})$ corresponding to a stationnary point of functional (4) will respect current continuity and albedo boundary conditions in addition to satisfying the diffusion equation.

- Conditions satisfied as a stationnary point of a fiunctional are said to be natural conditions.
- Conditions imposed to the trial functions, such as flux continuity or zero-flux boundary conditions, are said to be essential conditions.

The Lagrangian finite-element method 1

- The finite element method (FEM) is used for finding approximate solutions of partial differential equations. It is based on an expansion of the dependent variable(s), the particle flux in our case, into a linear combination of polynomial trial functions defined over subvolumes.
- The trial functions space must be chosen so as to ensure that improvement in the numerical approximation occurs with increase in the number I of subvolumes and/or with the degree K of the polynomial trial functions.
- The trial functions are known a priori and the corresponding coefficients can be found using a weighted residual approach or a variational formulation
- We have chosen to present the variational formulation, as it brings two important benefits:
  1. the intrinsic symmetry of the one-speed diffusion equation is always preserved by the discretization process,
  2. the boundary conditions are introduced in a consistent way.

The Lagrangian finite-element method 2

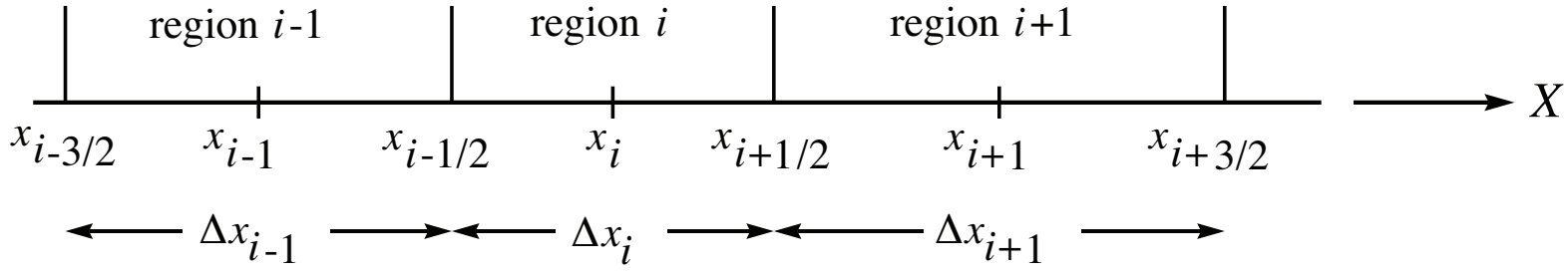
The polynomial basis $\{u_m(\boldsymbol{r})\,;\ m=1,M\}$ with $u_m(\boldsymbol{r}) \in W_1(V) \cap \mathcal{D}(V)$ is used to span the neutron flux $\phi(\boldsymbol{r})$ and flux variation $\delta\phi(\boldsymbol{r})$ over V, according to

(13) $$\phi(\boldsymbol{r}) = \sum_{m=1}^{M} \phi_m\, u_m(\boldsymbol{r}) \ \text{ and } \ \delta\phi(\boldsymbol{r}) = \sum_{m=1}^{M} \delta\phi_m\, u_m(\boldsymbol{r})$$

where the set of variational coefficients $\{\phi_m\,;\ m=1,M\}$ represents the unknown vector or the numerical solution to be obtained by the FEM.

- The FEM can be applied to various types and form of subvolumes or elements. Cartesian and hexagonal elements are the most widely used in reactor physics for full-core calculations.
- A Cartesian domain is first partitioned into rectangular parallelepipeds over which the nuclear properties are assumed to be uniform.
- A polynomial basis is defined over each element by using full tensorial products of 1D polynomials up to a given order.
- In the case of a primal variational formulation, Lagrange polynomials are chosen as polynomial basis in order to satisfy the requirement that $u_m(\boldsymbol{r}) \in W_1(V) \cap \mathcal{D}(V)$.

The Lagrangian finite-element method 3



We will consider a 1D Cartesian domain with I subvolumes Trial functions in space are transformed from the global coordinate x defined over element i with $x_{i-1/2} \leq x \leq x_{i+1/2}$ to local coordinate u defined over the unit domain, with $-1/2 \leq u \leq 1/2$. The following change of variable will be used:

(14)
$$u = \frac{1}{\Delta x_i}\left[x - \frac{1}{2}\left(x_{i-1/2} + x_{i+1/2}\right)\right]$$

where

(15)
$$\Delta x_i = x_{i+1/2} - x_{i-1/2} \quad .$$

The Lagrangian finite-element method 4

The polynomial trial functions are written

(16) $$u_m(x) = \sum_{i=1}^{I} \sum_{k=0}^{K} \delta_{i,k}^{m} L_k(u) \; ; \quad m = 1, M$$

where I is the total number of elements and $\delta_{i,k}^{m}$ is the finite element delta function, equal to 1 if the local unknown k in element i correspond to the global unknown m, and 0 otherwise.

- Each element i serves as support for a local basis of order–K Lagrange polynomials.
- These polynomial are defined so as to preserve the continuity of the global trial functions, as required by our primal variational formulation.

Linear Lagrange polynomials ($K = 1$) 1

This local basis contains two linear Lagrange polynomials in 1D, defined for local base points $u_0 = -1/2$ and $u_1 = 1/2$. They are written

(17) $$L_0(u) = \frac{1}{2} - u \quad \text{and} \quad L_1(u) = \frac{1}{2} + u$$

so that $L_0(-1/2) = L_1(1/2) = 1$ and $L_0(1/2) = L_1(-1/2) = 0$.

In this case, the variational coefficients ϕ_m are the flux values at abscissas $x_{i-1/2}$ and $x_{i+1/2}$. Continuity of the trial functions is preserved if each coefficient ϕ_m, corresponding to an internal mesh, is shared by two elements. For example, if a domain has left and right albedo boundary conditions, then $\delta_{i,k}^m$ is defined as

(18) $$\delta_{i,k}^m = \begin{cases} 1 & \text{if } m = i + k \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, if a domain has left and right zero-flux boundary conditions, then $\delta_{i,k}^m$ is defined as

(19) $$\delta_{i,k}^m = \begin{cases} 0 & \text{if } i = 1 \text{ and } k = 0 \\ 0 & \text{if } i = I \text{ and } k = 1 \\ 1 & \text{if } m = i + k - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Parabolic Lagrange polynomials ($K = 2$) 1

This local basis contains three parabolic Lagrange polynomials in 1D. They are written

(20) $$L_0(u) = -\frac{1}{8} - u + \frac{5u^2}{2}\ , \quad L_1(u) = \frac{5}{4} - 5u^2 \quad \text{and} \quad L_2(u) = -\frac{1}{8} + u + \frac{5u^2}{2}\ .$$

Again, the continuity of trial functions will be satisfied if some coefficients ϕ_m are shared by two elements. For example, if a domain has left and right albedo boundary conditions, then $\delta_{i,k}^m$ is defined as

(21) $$\delta_{i,k}^m = \begin{cases} 1 & \text{if } m = 2i + k - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, if a domain has left and right zero-flux boundary conditions, then $\delta_{i,k}^m$ is defined as

(22) $$\delta_{i,k}^m = \begin{cases} 0 & \text{if } i = 1 \text{ and } k = 0 \\ 0 & \text{if } i = I \text{ and } k = 2 \\ 1 & \text{if } m = 2i + k - 2 \\ 0 & \text{otherwise.} \end{cases}$$

Cubic Lagrange polynomials ($K = 3$) 1

This local basis contains four parabolic Lagrange polynomials in 1D. They are written

$$L_0(u) = -\frac{1}{8} + \frac{3u}{4} + \frac{5u^2}{2} - 7u^3 \ , \quad L_1(u) = \frac{5}{8} - \frac{5\sqrt{7}u}{4} - \frac{5u^2}{2} + 5\sqrt{7}u^3 \ ,$$

(23) $$L_2(u) = \frac{5}{8} + \frac{5\sqrt{7}u}{4} - \frac{5u^2}{2} - 5\sqrt{7}u^3 \quad \text{and} \quad L_3(u) = -\frac{1}{8} - \frac{3u}{4} + \frac{5u^2}{2} + 7u^3 \ .$$

Again, the continuity of trial functions will be satisfied if some coefficients ϕ_m are shared by two elements. For example, if a domain has left and right albedo boundary conditions, then $\delta_{i,k}^m$ is defined as

(24) $$\delta_{i,k}^m = \begin{cases} 1 & \text{if } m = 3i + k - 2 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, if a domain has left and right zero-flux boundary conditions, then $\delta_{i,k}^m$ is defined as

(25) $$\delta_{i,k}^m = \begin{cases} 0 & \text{if } i = 1 \text{ and } k = 0 \\ 0 & \text{if } i = I \text{ and } k = 3 \\ 1 & \text{if } m = 3i + k - 3 \\ 0 & \text{otherwise.} \end{cases}$$

Assembly of matrix system

Having defined our space of polynomial trial functions, we need to find the variational coefficients corresponding to the solution of the diffusion equation. Using the variational approach, we first rewrite Eq. (6) for the particular case of a Cartesian 1D domain. We have

$$\sum_{i=1}^{I} \int_{x_{i-1/2}}^{x_{i+1/2}} dx \; [D_i \, \nabla \delta\phi(x) \cdot \nabla \phi(x) + \Sigma_{\mathrm{r},i} \, \delta\phi(x) \, \phi(x) - \delta\phi(x) \, Q^{\diamond}(x)]$$

$$+ \mathcal{B}_- \, \delta\phi(x_{1/2}) \, \phi(x_{1/2}) \; + \; \mathcal{B}_+ \, \delta\phi(x_{I+1/2}) \, \phi(x_{I+1/2}) \; = \; 0 \tag{26}$$

$\forall \delta\phi(x) \in W_1(V) \cap \mathcal{D}(V)$. The two boundary terms $\mathcal{B}_-$ and $\mathcal{B}_+$ are defined as

$$\mathcal{B}_- = \begin{cases} \frac{1}{2} \frac{1-\beta_-}{1+\beta_-} & \text{if the domain has a left albedo condition} \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

and

$$\mathcal{B}_+ = \begin{cases} \frac{1}{2} \frac{1-\beta_+}{1+\beta_+} & \text{if the domain has a right albedo condition} \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

Assembly of matrix system

We next define a linear product and two different types of bilinear products:

(29) $$\langle Q^{\diamond}\, u_m \rangle = \sum_{i=1}^{I} \int_{x_{i-1/2}}^{x_{i+1/2}} dx\, Q^{\diamond}(x)\, u_m(x)$$

(30) $$\langle \nabla u_m, D \nabla u_n \rangle = \sum_{i=1}^{I} \int_{x_{i-1/2}}^{x_{i+1/2}} dx\, D_i\, \nabla u_m(x) \cdot \nabla u_n(x)$$

and

(31) $$\langle u_m, \Sigma\, u_n \rangle = \sum_{i=1}^{I} \int_{x_{i-1/2}}^{x_{i+1/2}} dx\, \Sigma_i\, u_m(x)\, u_n(x) \quad .$$

These bilinear products can be expressed for a Lagrangian FEM as

(32) $$\langle \nabla u_m, D \nabla u_n \rangle = \sum_{i=1}^{I} \frac{D_i}{\Delta x_i} \sum_{k=0}^{K} \sum_{\ell=0}^{K} \delta_{i,k}^{m}\, \delta_{i,\ell}^{n}\, Q_{k,\ell}$$

(33) $$\text{and} \quad \langle u_m, \Sigma\, u_n \rangle = \sum_{i=1}^{I} \Delta x_i\, \Sigma_i \sum_{k=0}^{K} \sum_{\ell=0}^{K} \delta_{i,k}^{m}\, \delta_{i,\ell}^{n}\, M_{k,\ell} \quad .$$

Assembly of matrix system

Substitution of Eqs. (29) to (31) into Eq. (26) leads to the discretized linear system, corresponding to the one-speed neutron diffusion equation. We write

$$\left\{ \sum_{m=1}^{M} \phi_m \left[\langle \nabla u_m, D \nabla u_n \rangle + \langle u_m, \Sigma_{\mathrm{r}}\, u_n \rangle \right] \right\} + \phi_1\, \mathcal{B}_-\, \delta_{n,1} + \phi_M\, \mathcal{B}_+\, \delta_{n,M} = \langle Q^{\diamond}\, u_n \rangle \tag{34}$$

where $n = 1, M$.

The system (34) is a linear matrix system of the form $\mathbb{A}\boldsymbol{\Phi} = \boldsymbol{Q}$ where the coefficient matrix $\mathbb{A}$ is symmetric, positive definite and diagonally dominant.

Equations (32) and (33) are written in term of the FEM mass matrix $\{M_{k,\ell}\,,\ k = 0, K \text{ and } \ell = 0, K\}$ and stiffness matrix $\{Q_{k,\ell}\,,\ k = 0, K \text{ and } \ell = 0, K\}$, defined as

$$M_{k,\ell} = \int_{-1/2}^{1/2} du\, L_k(u)\, L_\ell(u) \tag{35}$$

and

$$Q_{k,\ell} = \int_{-1/2}^{1/2} du\, \frac{d}{du} L_k(u)\, \frac{d}{du} L_\ell(u) \quad . \tag{36}$$

Unit matrices

These unit matrices can be integrated analytically, leading to

Linear Lagrange polynomials ($K = 1$):

$$\mathbb{M}_1 = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad \mathbb{Q}_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{37}$$

Parabolic Lagrange polynomials ($K = 2$):

$$\mathbb{M}_2 = \begin{bmatrix} \frac{1}{8} & 0 & -\frac{1}{24} \\ 0 & \frac{5}{6} & 0 \\ -\frac{1}{24} & 0 & \frac{1}{8} \end{bmatrix} \quad \text{and} \quad \mathbb{Q}_2 = \begin{bmatrix} \frac{37}{12} & -\frac{25}{6} & \frac{13}{12} \\ -\frac{25}{6} & \frac{25}{3} & -\frac{25}{6} \\ \frac{13}{12} & -\frac{25}{6} & \frac{37}{12} \end{bmatrix} \tag{38}$$

Cubic Lagrange polynomials ($K = 3$):

$$\mathbb{M}_3 = \begin{bmatrix} \frac{1}{15} & 0 & 0 & \frac{1}{60} \\ 0 & \frac{5}{12} & 0 & 0 \\ 0 & 0 & \frac{5}{12} & 0 \\ \frac{1}{60} & 0 & 0 & \frac{1}{15} \end{bmatrix} \quad \text{and} \quad \mathbb{Q}_3 = \begin{bmatrix} \frac{83}{15} & -\frac{21\sqrt{7}+25}{12} & \frac{21\sqrt{7}-25}{12} & -\frac{41}{30} \\ -\frac{21\sqrt{7}+25}{12} & \frac{65}{6} & -\frac{20}{3} & \frac{21\sqrt{7}-25}{12} \\ \frac{21\sqrt{7}-25}{12} & -\frac{20}{3} & \frac{65}{6} & -\frac{21\sqrt{7}+25}{12} \\ -\frac{41}{30} & \frac{21\sqrt{7}-25}{12} & -\frac{21\sqrt{7}+25}{12} & \frac{83}{15} \end{bmatrix} \tag{39}$$

Further remarks

We have presented the FEM in its simplest implementation.

- The Lagrangian formulation can be modified by using numerical integration to obtain the mass and stiffness matrices of Eqs. (35) and (36).
  - Using a Gauss-Lobatto quadrature with linear Lagrange polynomials produces a numerical solution that is equivalent to the mesh-corner finite difference method.
  - Using a Gauss-Legendre quadrature leads to superconvergent approximations.
- Discretization of 2D and 3D domains with a Lagrangian FEM produces a matrix system that is not compatible with the alternating direction implicit (ADI) method. This incompatibility is due to the off-diagonal term present in the mass matrices of Eqs. (37) to (39). This off-diagonal term can be suppressed by using a Gauss-Lobatto quadrature, leading to the primal finite-element approximations of the TRIVAC code.
- The Raviart-Thomas finite element method introduces an important class of approximations, based on a mixed-dual variational formulation.
  - A linear Raviart-Thomas finite element method, with Gauss-Lobatto integration of the unit matrices, is equivalent to the mesh-centered finite difference method.
  - All Raviart-Thomas FEMs are producing approximations that are compatible with the ADI method and with the TRIVAC code.