Cross sections

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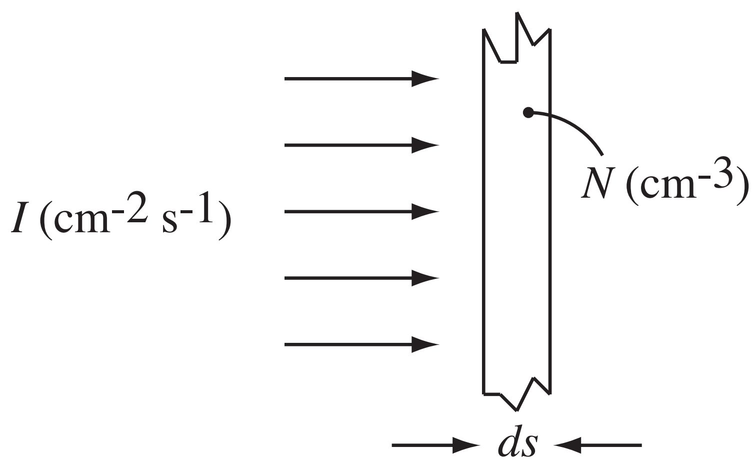
Content (week 4)

- Definition of cross sections
- Formation of a compound nucleus
  - selection rules
  - resonances
  - Porter and Thomas distribution
  - Wigner distribution
- The single level Breit-Wigner (SLBW) formula
  - resonance parameters of ^{232}Th
  - ^{235}U, ^{238}U and ^{239}Pu cross sections
  - low-energy variation of cross sections

Definition of cross sections

1. The trajectory of a neutron in a material is a straight line which can be interrupted by a nuclear interaction with a nucleus of the material.
2. The neutron-nucleus collision can result in a variety of nuclear reactions (such as the elastic scattering reaction).
3. The concept of cross section is used to describe the probability of each type of nuclear reaction. The probability for a neutron located at $\boldsymbol{r}$ and moving in a material at velocity V_{n} to undergo a nuclear reaction in a differential element of trajectory ds is independent of the past history of the neutron and is proportional to ds.

Let's consider a one-speed and parallel beam of neutrons of intensity I neutrons per unit surface and unit time. The beam hits perpendicularly a target of width ds. The target contains a unique type of nuclide with a number density of N nuclei per unit volume.



Definition of cross sections

The number density of nucleus in the target is given by

(1) $$N = \frac{\rho\, A_0}{M}$$

where ρ is the density (g/cm^3), M is the atomic mass of one nuclide (u) and A_0 is the Avogadro number defined as 6.022094×10^{23} u/g.

The intensity I if the beam is obtained from

(2) $$I = V_{\mathrm{R}}\, n$$

where V_{R} is the relative velocity of the neutrons with respect to the target and n is the number density of the neutrons (cm^{-3}) in the beam. If the target is at 0K, $V_{\mathrm{R}} = V_{\mathrm{n}}$.

The surfacic reaction rate dR_x is defined as the number of nuclear reactions of type x per unit time and unit surface of the target. It is experimentally found that dR_x is proportional to the number density N, to the intensity of the beam I, and to the width of the target, at the limit of zero width. The microscopic cross section σ_x is defined as the proportionality factor:

(3) $$dR_x = \sigma_x\, N\, I\, ds \quad .$$

Definition of cross sections

The microscopic cross section must have the dimension of a surface to make Eq. (3) dimensionally consistent. They are generally expressed in barn (b), with 1 b $= 10^{-24}$ cm^2. It is also common to define the macroscopic cross section Σ_x to group all the characteristics of the target in a single value. It is defined as $\Sigma_x = N\,\sigma_x$.

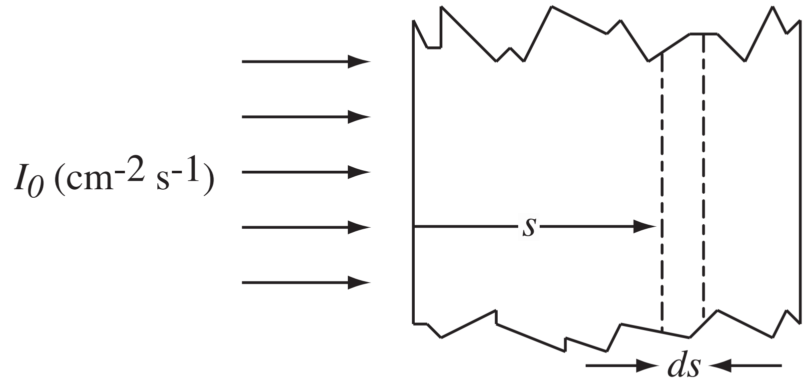
If the material of the target is an homogeneous mixture of different types i of nuclides, the resulting macroscopic cross section is

$$\Sigma_x = \sum_i N_i\,\sigma_{x,i} \quad . \tag{4}$$

Moreover, we define the total macroscopic cross section as the sum of cross sections from all nuclear reactions. We write

$$\Sigma = \sum_x \Sigma_x \quad . \tag{5}$$

Definition of cross sections



Let's consider now a one-speed and parallel beam of neutrons of intensity I_0 neutrons per unit surface and unit time. The beam hits perpendicularly a slab of finite width. The collision or total reaction rate dR is the number of collisions of neutrons in an elemental width ds located at distance s, per unit time and unit surface of the slab. Each time a neutron collide in ds, it is removed from the uncollided beam. The reaction rate is written

$$dR = -dI(s) = I(s)\,\Sigma\,ds \tag{6}$$

where $I(s)$ is the intensity of the uncollided beam after a distance s in the slab. Integrating this equation between 0 and s, we obtain

$$I(s) = I_0\,\mathrm{e}^{-\Sigma s} \quad . \tag{7}$$

Definition of cross sections 5

The ratio $I(s)/I_0$ is the probability for the neutron to be uncollided after a distance s. The probability of an interaction for this neutron in the following elemental width ds is $\Sigma\, ds$. The probability $P(s)\, ds$ for a neutron in I_0 to collide in ds is therefore

$$P(s)\, ds = \Sigma\, \mathrm{e}^{-\Sigma s}\, ds \quad . \tag{8}$$

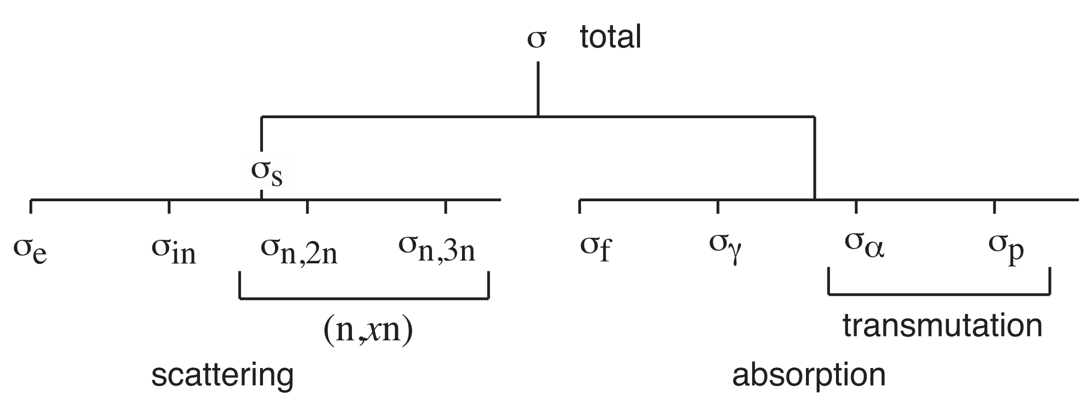
This equation is useful to define the mean free path λ of neutrons in an infinite slab, an important quantity in reactor physics. The mean free path is the average trajectory length of the neutrons in an infinite and homogeneous material. This value is obtained from equation

$$\lambda = \int_0^\infty ds\, s\, P(s) = \frac{1}{\Sigma} \quad . \tag{9}$$

Definition of cross sections

1. Each type of nuclear reaction is characterized by a specific microscopic cross section σ_x which is also function of the type of nuclide and on the velocity or kinetic energy of the neutron. However, the potential cross section is independent of neutron energy.
2. An elastic neutron-nucleus collision is characterized by an elastic scattering cross section $\sigma_{\rm e}$. This reaction includes both potential and resonant elastic collisions.
3. The symbol Q represents the energy produced by a nuclear reaction in the form of kinetic energy in excess of $e_{\rm exc}$. In a threshold reactions, Q is negative and the nuclear reaction can occur only if $e_{\rm exc} \geq -Q$. If a scattering reaction has a threshold energy $-Q$, the compound nucleus is left excited after the collision and decay with gamma ray emission of energy $-Q$ (short half life). This is the inelastic scattering cross section ($\sigma_{\rm in}$).

The scattering cross section $\sigma_{\rm s}$ is written $\sigma_{\rm s} = \sigma_{\rm e} + \sigma_{\rm in} + \sum_{x \geq 2} \sigma_{{\rm n},x{\rm n}}$.



Definition of cross sections

In a similar way, we may define cross sections for reactions involving the absorption of the incident neutron. A representation of this hierarchy is given in figure.

Nuclear reactions in reactor physics are often represented with the following notation:

(n,n):	elastic scattering,
(n,n'):	inelastic scattering,
(n,γ):	the radiative capture cross section σ_γ
(n,f):	the fission cross section σ_f,
(n,α), (n,p):	transmutation cross sections σ_α (an α particle is emitted)
	or σ_p (a proton is emitted)
(n,2n):	n-2n reaction, etc.

All these reactions, with the exception of the potential scattering, involve the formation of a compound nucleus and are characterized by cross sections that may exhibit high variation with neutron energy. This phenomena will be studied in the next section.

The total cross section is the sum of all existing cross sections.

Formation of a compound nucleus

Any nuclear reaction with formation of a compound nucleus are proceeding in two successive steps:

1. The incident neutron is absorbed by the target nucleus ${}^{A}_{Z}X$ to form an excited state of nucleus ${}^{A+1}_{Z}X$ called the compound nucleus. During the formation process, the available kinetic energy of the incident neutron $e_{\rm exc}$ and the binding energy of this additional neutron $S_{\rm n}(A+1,Z)$ are distributed among all the nucleons of the compound nucleus. The binding energy is computed in term of the mass default using

   $$S_{\rm n}(A+1,Z) = \left[M(A,Z) + m_{\rm n} - M(A+1,Z)\right]c^2 \tag{10}$$

   where $M(A,Z)$ is the mass of isotope ${}^{A}_{Z}X$ expressed in u, $m_{\rm n} = 1.008665\,{\rm u}$ is the mass of the neutron and $c^2 = 931.5\,{\rm MeV}/{\rm u}$ is the square of the light velocity in void.

2. The compound nucleus decay with a half life between 10^{-22} and 10^{-14} second, without reminding how it was formed. The neutronic reactions are not the only ones that can produce a compound nucleus; a photo-nuclear interaction (with an incident γ ray) can produce the same effect. This decay can occur following a number of decay channels, as represented in table. Each decay channel involves the production of secondary rays and/or particles which can emerge in ground or excited state.

Formation of a compound nucleus

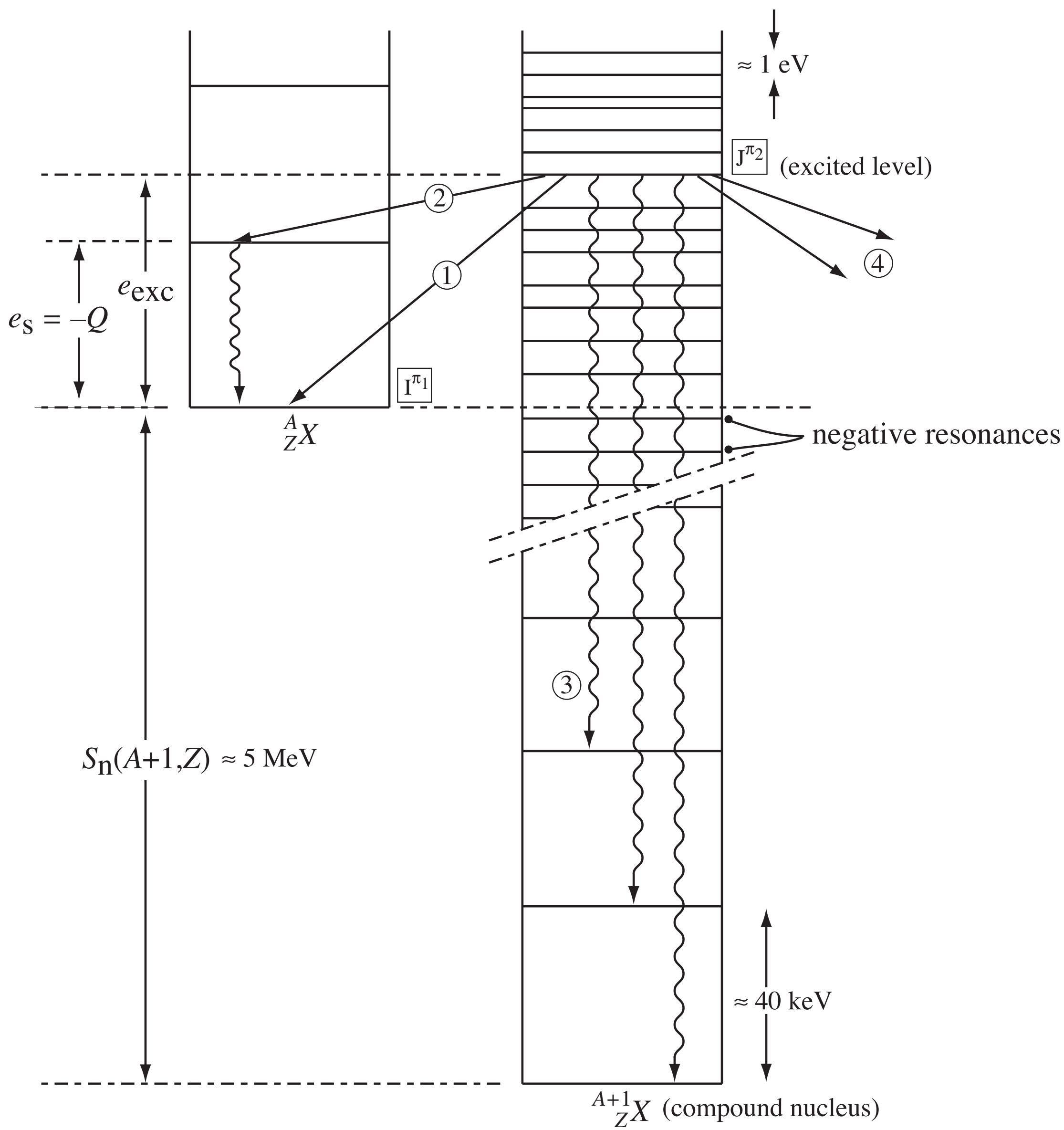
	Type of reaction	Number of decay channels
①	elastic scattering	1 channel
②	inelastic scattering	1 channel if $e_{\rm exc} \geq -Q$
③	radiative capture	very high number of channels
④	fission	$\simeq$ 2 or 3 channels if $e_{\rm exc} + S_{\rm n} \geq e_{\rm f}$

The compound nucleus model consists to write the interaction as a two-stage reaction,

$$ {}^{A}_{Z}X + {}^{1}_{0}n \rightarrow {}^{A+1}_{\;\;Z}X^{*} \rightarrow \begin{cases} ① \\ ② \\ ③ \\ ④ \end{cases} , \qquad (11)$$

where the asterisk indicates that the compound nucleus is in an excited state. The first arrow denotes the formation stage and the second, the decay stage. The energy-level diagram for this reaction is shown in figure.

Formation of a compound nucleus



Formation of a compound nucleus 4

A decay channel is open only if the three following laws are observed: conservation of energy, linear and angular momentum of the nuclide-nucleus pair. The first two laws have already been studied in Week 3. The conservation of angular momentum is written

$$\boldsymbol{J} = \boldsymbol{I} + \boldsymbol{K} + \boldsymbol{L} \quad \text{where} \tag{12}$$

$\boldsymbol{I}\,,\;\boldsymbol{K} =$ angular momentum of the target nucleus ${}^{A}_{Z}X$ in its ground state and of the neutron

$\boldsymbol{J} =$ angular momentum of the compound nucleus ${}^{A+1}_{\;\;Z}X$ in its excited state

$\boldsymbol{L} =$ orbital angular momentum of the nuclide-nucleus pair in the CM.

The classical definition of the orbital angular momentum of the nuclide-nucleus pair about the origin of the CM is $\boldsymbol{L} = m(\boldsymbol{r}_{\rm n} - \boldsymbol{r}_{\rm CM}) \times \boldsymbol{v}_{\rm n} + mA(\boldsymbol{r}_{\rm A} - \boldsymbol{r}_{\rm CM}) \times \boldsymbol{v}_{\rm A}$.

Using relations in week 3 and defining the $\boldsymbol{e}_{\rm z}$–directed axis as perpendicular to the plane where the particles are moving, this equation simplifies to

$$\boldsymbol{L} = \Delta R\, V_{\rm R}\, m_0\, \boldsymbol{e}_{\rm z} \tag{13}$$

where ΔR is the distance between the lines of definition of vectors $\boldsymbol{v}_{\rm n}$ and $\boldsymbol{v}_{\rm A}$ in the CM and m_0 is the reduced mass of the neutron defined as $m_0 = \frac{m\,A}{A+1}$.

Formation of a compound nucleus

In reactor physics, the principle of conservation of angular momentum involves principles of quantum mechanics. There is a relation between the orbital angular momentum of the neutron-nucleus pair and the angular momentum quantum number.

In quantum mechanics, the modulus of the orbital angular momentum vector $\boldsymbol{L}$ is related to the angular momentum quantum number ℓ, a positive integer, by the relation

$$L = \sqrt{\ell(\ell+1)}\,\hbar \tag{14}$$

where $\hbar = 1.054494 \times 10^{-34}$ J·s is the reduced Plank constant. A comparison of the classical expression of L in Eq. (13) and of the quantum expression in Eq (14) indicates that $\ell = 0$ is likely to occur if V_{R} is low. However in cases where ΔR or V_{R} is high, greater values of ℓ become possible. Note that high values of ΔR is only possible when the target nucleus has a large radius, a characteristics of heavy nuclides. The following nomenclature is universally accepted:

$$\ell = \begin{cases} 0; & \text{s–wave interaction,} \\ 1; & \text{p–wave interaction,} \\ 2; & \text{d–wave interaction.} \end{cases}$$

In conclusion, s–wave interactions are the most common for low-energy incident neutrons and p– or d–wave interactions will be more probable for heavy target nuclides.

Selection rules

The parity π is also required. It is a binary quantity (equal to +1 or -1) that characterize the energy levels of the particles in interaction. The basic information about parity is:

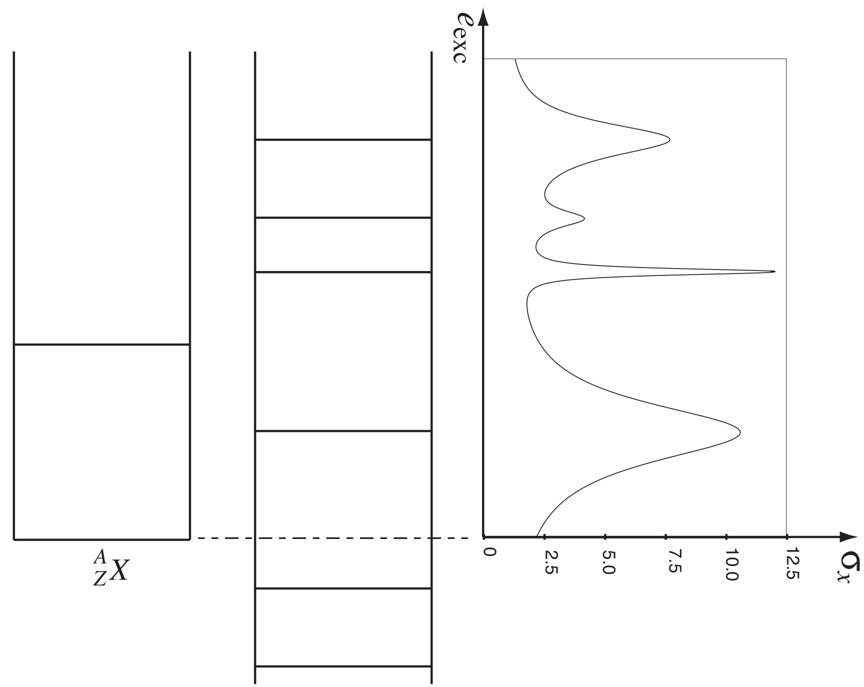
	Spin (I, J or K)	Parity (π)
Neutron	1/2	+1
Nucleus (ground state):		
even A and even Z	0	+1
even A and odd Z	integer and $\neq 0$	unknown
odd A	half integer	unknown

Notations I^{π_1} and J^{π_2} are often used to represent the spin and parity of initial and final levels, respectively. An energy level of the compound nucleus can be excited if the following two selection rules are fulfill (see ENDF–102):

1. the spin J of the excited level in ${}^{A+1}_{Z}X$ is an element of set $\{J_{\rm min}, J_{\rm min}+1, \ldots, J_{\rm max}\}$ where $J_{\rm min} = \left| |I-\ell| - \frac{1}{2} \right|$ and $J_{\rm max} = \ell + I + \frac{1}{2}$,
2. the parity of the excited level in ${}^{A+1}_{Z}X$ must obey $\pi_2 = \pi_1(-1)^\ell$ where π_1 and π_2 are the parity of the target (ground level) and of the compound nucleus (excited level).

Resonances

Notice that $e^* = S_\mathrm{n}(A+1,Z) + e_\mathrm{exc}$ is the available excitation energy of the compound nucleus and is measured on the CM. If the compound nucleus has an excited state at e_i that is close to e^*, then one can have resonance condition and the compound nucleus formation cross section will show a peak at the neutron incident energy corresponding to e_i. These peaks are at the origin of the resonant cross section phenomena, as depicted in figure. We also observe that the resonance peaks for a cross section of isotope ${}^{A}_{Z}X$ correspond to the excitation states of isotope ${}^{A+1}_{Z}X$ located above energy $S_\mathrm{n}(A+1,Z)$.



Resonances

1. Each excitation level e_i has a certain energy width γ_i, due to the Heisenberg uncertainty principle. The width γ_i corresponds to its finite lifetime τ_i:

   $$\gamma_i = \frac{\hbar}{\tau_i} \quad . \tag{15}$$

   The finite lifetime is the average lifetime of the compound nucleus at level i. It is the reciprocal of the radioactive decay constant. The smaller the width means the longer the lifetime of the level. The energy width γ_i is defined in the CM.

2. The resonance width is related to the formation of the compound nucleus at level i. The probability for it to decay with a nuclear reaction of type x is given in term of the partial resonance width $\gamma_{x,i}$ as

   $$P_{x,i} = \frac{\gamma_{x,i}}{\gamma_i} \tag{16}$$

   so that

   $$\gamma_i = \sum_x \gamma_{x,i} \quad . \tag{17}$$

   The decay channel x is one of the nuclear reactions. The partial width corresponding to the scattering reaction is the neutron width $\gamma_{\mathrm{n},i}$.

Resonances

3. The peak energy of a resonance may be negative, in the case where a level of the compound nucleus is close to the state corresponding to the capture of a neutron by the target but is located below this state (called a negative resonance).
4. Intermediate and heavy isotopes are characterized by a high number of resonances. It become possible to apply statistical studies on the resonance widths γ and of their spacing D, corresponding to a given spin and parity. This statistical treatment is useful for the representation of the unresolved resonances in domain $10\mathrm{keV} \leq E \leq 300\mathrm{keV}$.
5. The neutron width is a function of the incident neutron energy and is written $\gamma_{\mathrm{n}}(e_{\mathrm{exc}})$. The statistical treatment on the neutron width is applied on the random variable

$$x = \frac{\gamma_{\mathrm{n}}^{\ell}}{\langle \gamma_{\mathrm{n}}^{\ell} \rangle} \tag{18}$$

where the reduced neutron width $\gamma_{\mathrm{n}}^{\ell}$, a function of the angular momentum quantum number ℓ, is defined as

$$\gamma_{\mathrm{n}}^{\ell} = \frac{\gamma_{\mathrm{n}}(e_{\mathrm{exc}})}{v_{\ell}(e_{\mathrm{exc}})\sqrt{e_{\mathrm{exc}}}} \tag{19}$$

where $v_{\ell}(e_{\mathrm{exc}})$ is a coefficient related to the penetrability of the potential barrier in the target nucleus.

Resonances

The first three values of the penetrability coefficient are

$$
\begin{aligned}
v_0(e_{\text{exc}}) &= 1 \\
v_1(e_{\text{exc}}) &= \left(\frac{R}{\lambdabar}\right)^2 \left[1 + \left(\frac{R}{\lambdabar}\right)^2\right]^{-1} \\
v_2(e_{\text{exc}}) &= \left(\frac{R}{\lambdabar}\right)^4 \left[9 + 3\left(\frac{R}{\lambdabar}\right)^2 + \left(\frac{R}{\lambdabar}\right)^4\right]^{-1}
\end{aligned} \tag{20}
$$

where R is the hard sphere radius of the nucleus. It can be computed in term of the atomic mass ratio A using $R = \left(0.123\, A^{1/3} + 0.08\right) \times 10^{-12}\ \text{cm}$, and where $\lambdabar$ is the reduced wavelength of the neutron, defined by the relation

$$
\lambdabar = \frac{\hbar}{m_0\, V_{\text{R}}} = \frac{\hbar}{\sqrt{2 m_0\, e_{\text{exc}}}} \tag{21}
$$

where m_0 is the reduced mass of the neutron.

Resonances

The reduced wavelength can also be written in term of the neutron mass m and of the excitation energy $E_{\rm exc}$ in the LAB. We write

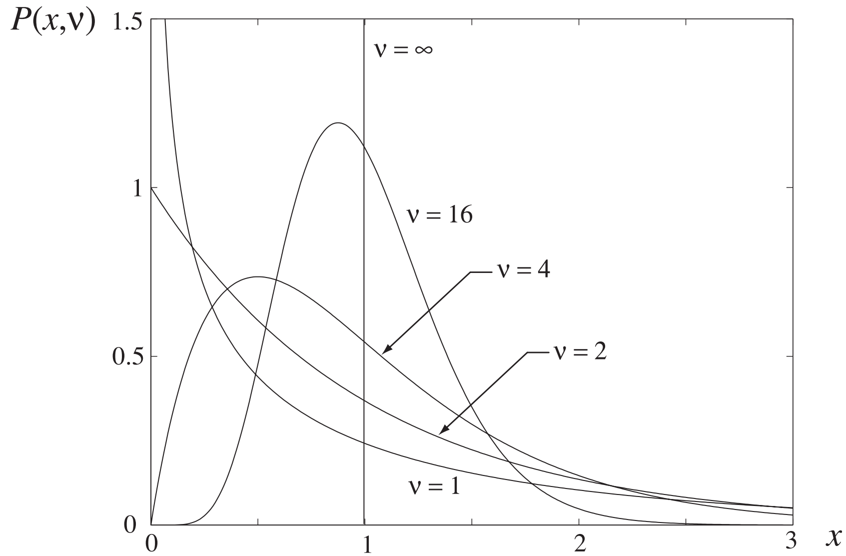
$$\lambdabar = \frac{A+1}{A} \frac{\hbar}{\sqrt{2m\,E_{\rm exc}}} = \left(0.4552136 \times 10^{-9}\right) \frac{A+1}{A} \frac{1}{\sqrt{E_{\rm exc}}} \ \ {\rm cm} \tag{22}$$

where

$$E_{\rm exc} = \frac{A+1}{A}\, e_{\rm exc} \tag{23}$$

is expressed in eV. Using the above definitions, we find that the reduced neutron width $\gamma_{\rm n}^{\ell}$ can be expressed in $\sqrt{\rm meV}$.

Porter and Thomas distribution



Porter and Thomas have shown that the resonance widths γ_γ, $\gamma_{\rm f}$ and the normalized neutron width $\gamma_{\rm n}^\ell$ corresponding to a single spin and parity obey a chi-square statistics:

$$P(x,\nu) = \frac{\nu}{2\mathcal{G}(\nu/2)} \left(\frac{\nu x}{2}\right)^{\frac{\nu}{2}-1} {\rm e}^{-\nu x/2} \tag{24}$$

where $x = \gamma_x / \langle \gamma_x \rangle$ is the random variable defined as the ratio of the actual resonance width divided by the average resonance width for reaction x. The random variable $x = \frac{\gamma_{\rm n}^\ell}{\langle \gamma_{\rm n}^\ell \rangle}$ must be used in the case of the neutron width.

Porter and Thomas distribution 2

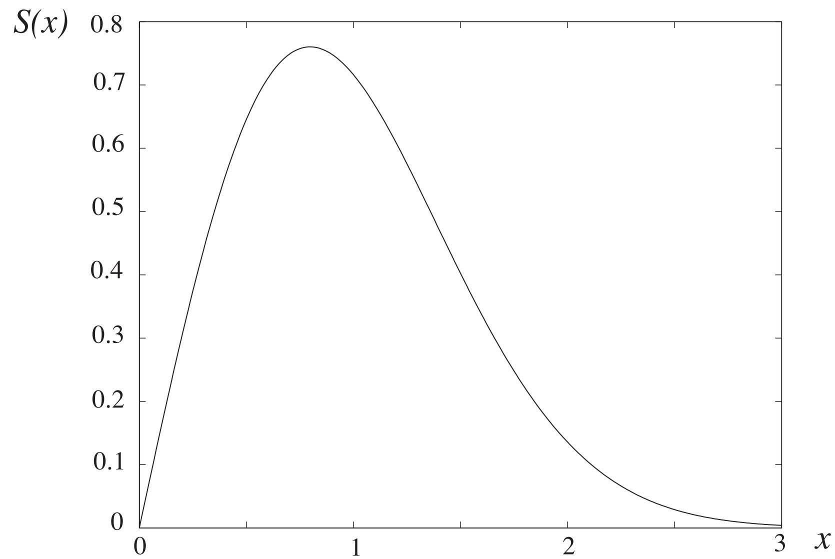
The number of degrees of freedom ν in the Porter-Thomas distribution is equal to the number of decay channels of the compound nucleus. The average value of x is 1 and its most probable value is $x_{\rm p} = 1 - 2/\nu$ if $\nu \geq 2$. In the case where $\nu = 1$, $P(x, 1)$ becomes infinite at $x = 0$. The gamma function $\mathcal{G}(\rho)$ is defined as

(25) $$\mathcal{G}(\rho) = \int_0^\infty dt\, \mathrm{e}^{-t}\, t^{\rho-1}$$

so that $\mathcal{G}(1/2) = \sqrt{\pi}$, $\mathcal{G}(1) = 1$ and $\mathcal{G}(\rho) = (\rho - 1)\, \mathcal{G}(\rho - 1)$. The Porter-Thomas is depicted in figure for different values of ν.

In the case of a scattering reaction, $\nu = 1$ and the Porter-Thomas distribution simplifies to a simple decreasing exponential. In the case of a radiative capture, the number of decay channels is very high and $P(x, \infty) = \delta(x - 1)$, the Dirac delta distribution. In this case, γ_γ will be almost constant from resonance to resonance.

Wigner distribution



The statistical distribution of the resonance spacings corresponding to a single spin and parity follows the Wigner distribution written as

$$S(x) = \frac{\pi}{2}\, x\, \mathrm{e}^{-\frac{\pi}{4} x^2} \quad \text{where} \quad x = \frac{D}{\langle D \rangle} \tag{26}$$

and depicted in figure.

The SLBW formula

The single level Breit and Wigner (SLBW) formulas result from the application of the Schrödinger equation to a compound nucleus model with a single excitation level. With this model, variations in cross sections are related to the width and energy characteristics of this excited level.

For reactions involving the absorption of the incident neutron, such as radiative capture or fission, the SLBW formula is written in term of the reduced energy variable

$$u = \frac{2}{\gamma_1}(e_{\rm exc} - e_1) \tag{27}$$

as

$$\sigma_x(e_{\rm exc}) = \sigma_0 \, \frac{\gamma_{x,1}}{\gamma_1} \, \frac{1}{1+u^2} \tag{28}$$

where

$\gamma_{x,1} =$ resonance width of the excited level for an absorption reaction of type x

$\gamma_1 =$ total resonance width of the excited level

$e_1 =$ energy of the excited level in the CM relative to the ground level of the target nucleus.

The SLBW formula 2

The parameter σ_0 is defined as

(29) $$\sigma_0 = 4\pi\lambdabar^2\, g_J\, \frac{\gamma_{\mathrm{n},1}(e_{\mathrm{exc}})}{\gamma_1} = \frac{2\pi\hbar^2}{m_0\,\sqrt{e_{\mathrm{exc}}}}\, v_\ell(e_{\mathrm{exc}})\, g_J\, \frac{\gamma^\ell_{\mathrm{n},1}}{\gamma_1}$$

where the expression for $\gamma_{\mathrm{n},1}(e_{\mathrm{exc}})$ and $\lambdabar^2$ were recovered from Eqs. (19) and (21), respectively. In this relation, $\gamma^\ell_{\mathrm{n},1}$ is constant. The statistical spin factor g_J is defined as

(30) $$g_J = \frac{2J+1}{(2I+1)(2K+1)} = \frac{2J+1}{2(2I+1)}$$

since $K = 1/2$ for the neutron. The excited level of the compound nucleus has a spin J function of the angular momentum quantum number ℓ and consistent with the selection rules.

The SLBW formula

In the case of the elastic scattering reaction, the cross section is the sum of three components. The first two are the potential and resonant cross section terms. There is also an interaction term between the first two components. The interference term arises in the quantum mechanical model when the modulus of the sum of two complex quantities is taken. The SLBW formula corresponding to the elastic scattering cross section is

$$\sigma_{\rm e}(e_{\rm exc}) = \sigma_{\rm p}^{\ell} + \sigma_0 \, \sin 2\phi_\ell \, \frac{u}{1+u^2} + \sigma_0 \left(\frac{\gamma_{\rm n,1}}{\gamma_1} - 2\sin^2\phi_\ell \right) \frac{1}{1+u^2} \quad . \tag{31}$$

The potential component $\sigma_{\rm p}^{\ell}$ is defined in term of the angular momentum quantum number ℓ as

$$\sigma_{\rm p}^{\ell} = 4\pi\lambda\!\!\!^{-2} \, (2\ell+1) \, \sin^2\phi_\ell \tag{32}$$

where the integer quantum number ℓ is ≥ 0 and where ϕ_ℓ are the shift factors.

The SLBW formula

The first values of the shift factor ϕ_ℓ are written as

$$\begin{aligned}\phi_0 &= \frac{a}{\lambda\!\!\!^{-}} \\ \phi_1 &= \frac{a}{\lambda\!\!\!^{-}} - \tan^{-1}\frac{a}{\lambda\!\!\!^{-}} \\ \phi_2 &= \frac{a}{\lambda\!\!\!^{-}} - \tan^{-1}\frac{\frac{3a}{\lambda\!\!\!^{-}}}{3-\left(\frac{a}{\lambda\!\!\!^{-}}\right)^2} \quad .\end{aligned} \tag{33}$$

We have used the diffusion radius a different from the hard sphere radius R introduced earlier. In general, the ratio $a/\lambda\!\!\!^{-}$ is small with respect to one, and the potential cross section component σ_p^0 for s wave interactions is almost equal to the classical value corresponding to a “billiard-ball" collision:

$$\sigma_\mathrm{p}^0 = 4\pi\, a^2 \quad . \tag{34}$$

The SLBW formula

Summation of Eq. (28) with Eqs. (31) corresponding to all absorption types of nuclear reactions leads to the SLBW expression of the total cross section:

$$\sigma(e_{\rm exc}) = \sigma_{\rm p}^{\ell} + \sigma_0 \, \sin 2\phi_\ell \, \frac{u}{1+u^2} + \sigma_0 \, \cos 2\phi_\ell \, \frac{1}{1+u^2} \quad . \tag{35}$$

Equations (28) and (31) are giving the variation of the absorption-type and scattering cross sections with energy $e_{\rm exc}$ in the case of a unique resonance. These variations are depicted in figure.

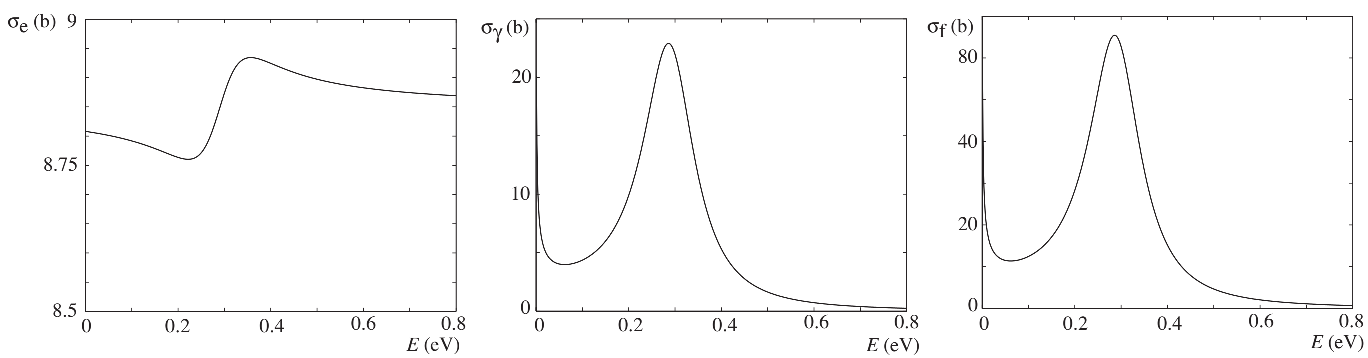
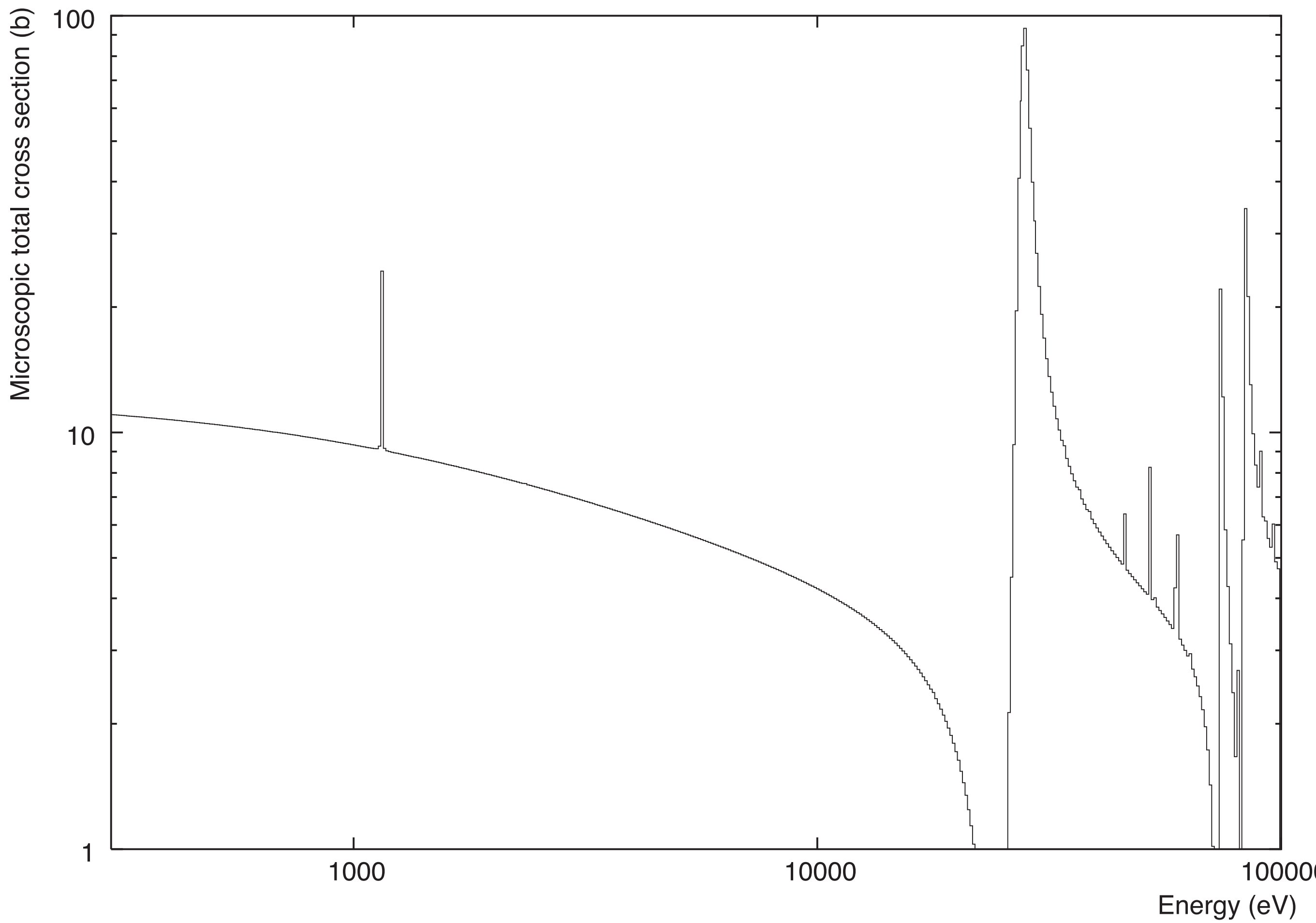


Figure depicts a nice illustration of the effect of the interference term in Eq. (31).

The SLBW formula

This is a very well known shielding issue associated with ^{56}Fe. The cross section is almost vanishing near 24 keV, so that incident neutrons with this energy are undergoing very few collisions in iron.



The SLBW formula

1. The SLBW formula can only be applied to the case of an isolated resonance. However, most isotopes feature many resonances at increasing energies. A crude approximation consists to neglect resonance interactions, so that the cross section of a reaction $x \neq e$ is written by summing the contributions of I resonances as

$$\sigma_x(e_{\rm exc}) = \sum_{i=1}^{I} \sigma_0 \, \frac{\gamma_{x,i}}{\gamma_i} \, \frac{1}{1+u^2} \quad . \tag{36}$$

   A similar expression can be written for $\sigma_{\rm e}(e_{\rm exc})$.

2. In the general case, many levels of the compound nucleus interact together and a multilevel formula is required. In this case, the cross section are obtained by summing the contributions from each individual resonance, taking care to introduce interaction terms in the sum.

3. At energies above $\simeq 10$ keV, resonances become unresolved and are represented by statistical parameters. At higher energies, the resonance become tighter up to the point where they overlap, leading to the continuum energy domain.

The SLBW formula

The SLBW equations (28) and (31) are function of the excitation energy $e_{\rm exc}$ in the CM. However, most theoretical developments used in reactor physics are expecting cross sections defined in term of LAB-related quantities. We remember the expression of the excitation energy $E_{\rm exc}$ in the LAB as

$$E_{\rm exc} = \frac{A+1}{A}\, e_{\rm exc} = \frac{1}{2} m V_{\rm R}^2 \quad . \tag{37}$$

In case where the target nuclide is initially at rest, $E_{\rm exc}$ is equal to the initial energy of the neutron in the LAB. If we replace $e_{\rm exc}$ by $E_{\rm exc}$ in Eqs. (28) and (31), these equations remain valid provided that we redifine LAB-related peak energy and resonance widths as

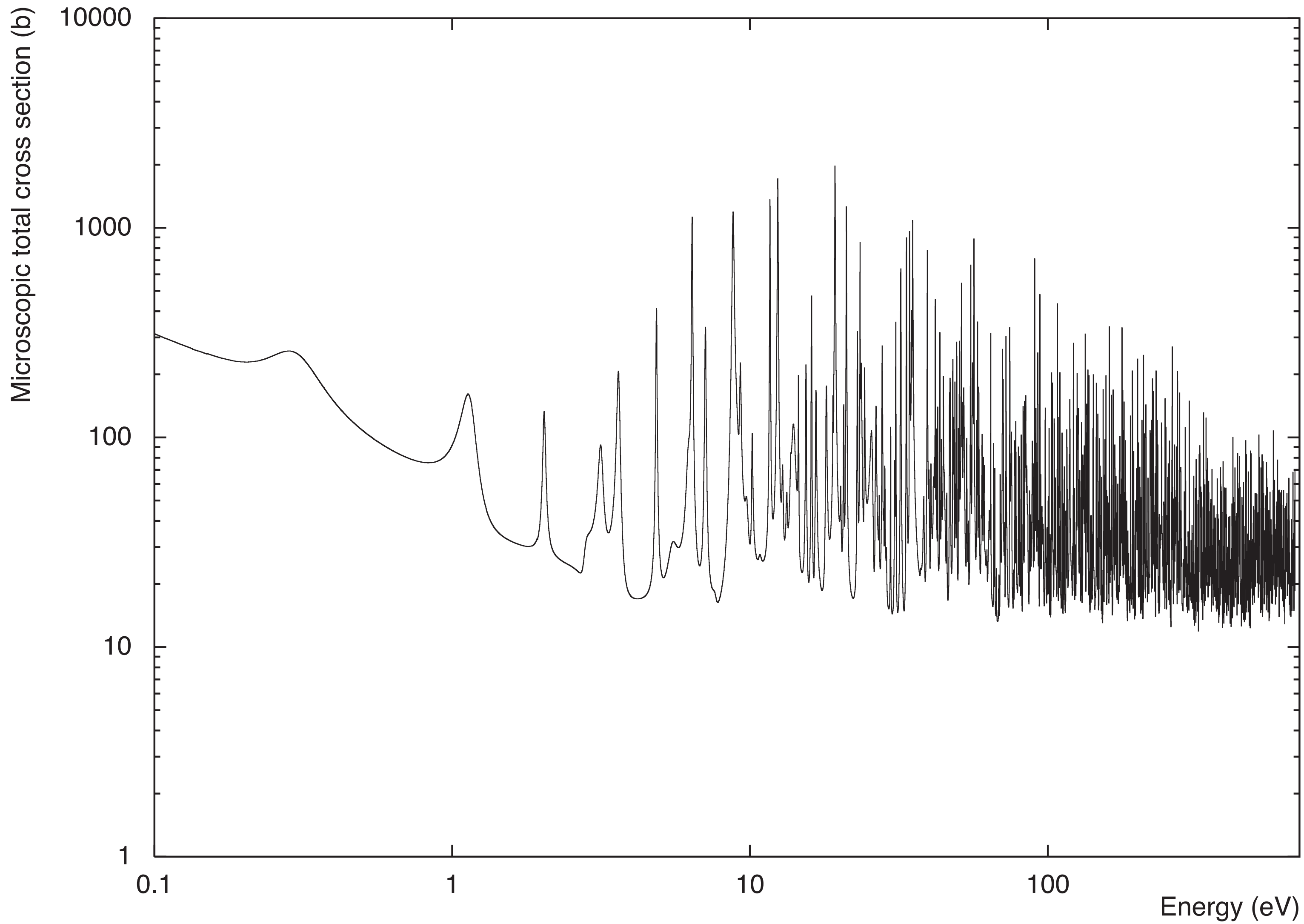
$$E_1 = \frac{A+1}{A}\, e_1 \quad \text{and} \quad \Gamma_{x,1} = \frac{A+1}{A}\, \gamma_{x,1} \quad . \tag{38}$$

Peak energy and resonance widths are measured and are reported in reference tables as LAB-defined values, similar to those defined in Eqs. (38). The neutron widths are reported with the statistical spin factor included, as $g_J \Gamma_{\rm n,1}(E_1)$.

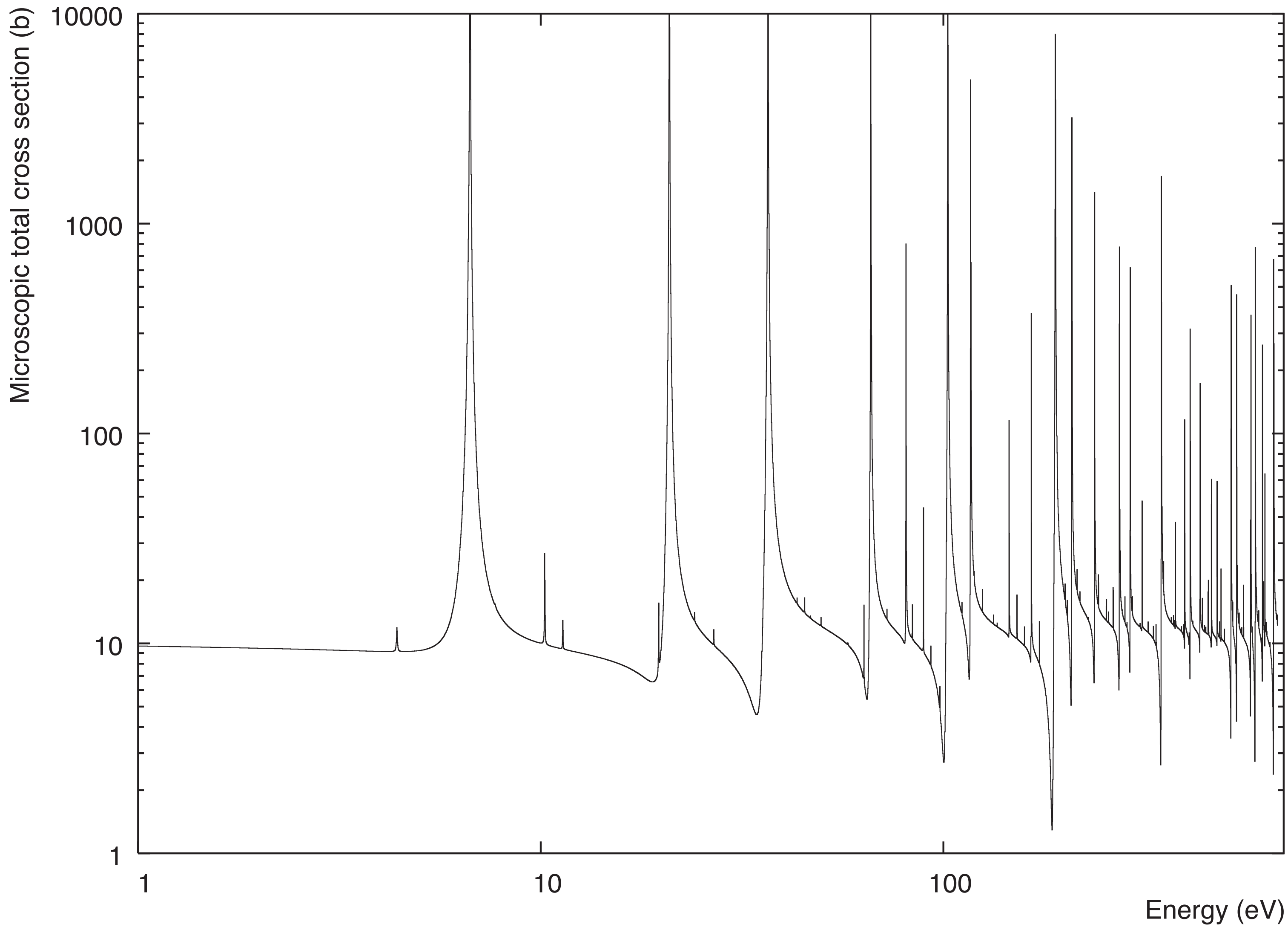
Resonance parameters of ^{232}Th

i	E_i (eV)	$g\Gamma_{\mathrm{n},i}(E_i)$ (meV)	$\Gamma_{\gamma,i}$ (meV)	ℓ
1	8.346	0.0003	29.0	1
2	13.111	0.0002		1
3	21.783	2.0200	24.5	0
4	23.439	3.8800	26.6	0
5	36.926	0.0010		1
6	38.165	0.0006		1
7	40.925	0.0006		1
8	47.001	0.0014		1
9	49.850	0.0006		1
10	54.130	0.0011		1
11	58.780	0.0096		1
12	59.514	3.9000	23.7	0
13	64.580	0.0005		1
14	69.224	43.8000	21.9	0

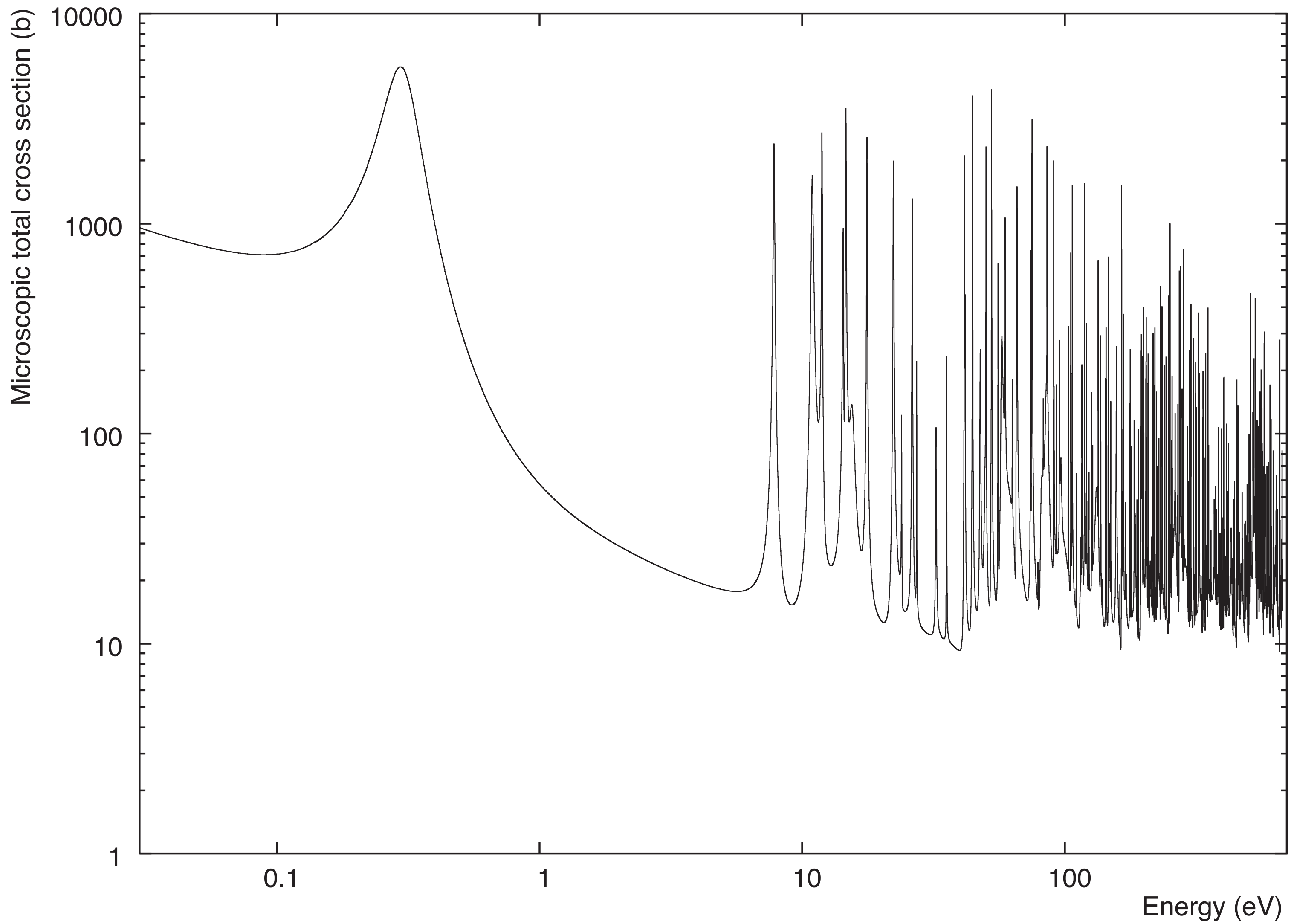
Uranium-235 total cross section



Uranium-238 total cross section



Plutonium-239 total cross section



Low-energy variation of cross sections 1

We consider a resonance located at an energy $e_{x,1}$ above the thermal energy domain (> 1 eV) and study its contribution at thermal energies. We will rewrite the SLBW formulas and take their limits as $e_{x,1}$ approaches zero.

If $R/\lambda\!\!\!^{-} << 1$ and $a/\lambda\!\!\!^{-} << 1$, only s wave interactions are contributing to the cross sections. Using the LAB variables, the SLBW Eqs. reduce to

(39) $$\sigma_x(E_{\rm exc}) = \sigma_0 \, \frac{\Gamma_{x,1}}{\Gamma_1} \, \frac{1}{1+u^2} \quad ,$$

(40) $$\sigma_{\rm e}(E_{\rm exc}) = 4\pi \, a^2 + \sigma_0 \, \frac{2a}{\lambda\!\!\!^{-}} \, \frac{u}{1+u^2} + \sigma_0 \, \frac{\Gamma_{\rm n,1}}{\Gamma_1} \, \frac{1}{1+u^2} \quad ,$$

(41) $$\sigma(E_{\rm exc}) = 4\pi \, a^2 + \sigma_0 \, \frac{2a}{\lambda\!\!\!^{-}} \, \frac{u}{1+u^2} + \sigma_0 \, \frac{1}{1+u^2} \quad ,$$

where the reduced variable u and parameter σ_0 are defined as

(42) $$u = \frac{2}{\Gamma_1} \left(E_{\rm exc} - E_1 \right) \quad \text{and} \quad \sigma_0 = 4\pi \lambda\!\!\!^{-2} \, g_J \, \frac{\Gamma_{\rm n,1}(E_{\rm exc})}{\Gamma_1} \quad .$$

Low-energy variation of cross sections 2

1. These equations are valid at energies $E \leq 300$ keV for a heavy nuclide and at energies $E \leq 1$ MeV for an intermediate nuclide ($A \simeq 40$).
2. We will next consider an isotope without negative resonance and without resonances in the thermal energy domain. In this case, the energy-variation of the cross sections at thermal energies is dictated by the energy variation of $\lambdabar$ and $\Gamma_{\mathrm{n},1}$. The absorption-type widths, such as $\Gamma_{\gamma,1}$ or $\Gamma_{\mathrm{f},1}$, are constant in energy.
3. The neutron width $\Gamma_{\mathrm{n},1}$, on the other hand, varies as $\sqrt{E}$ for a s wave interaction, as $E^{3/2}$ for a p wave interaction and as $E^{5/2}$ for a d wave interaction. The squared reduced wavelength $\lambdabar^2$ varies as $1/E$.
4. We therefore observe that s wave absorption-type reactions feature a characteristic $1/v$–dependence. At low energies, the probability of these interactions is directly proportional to the time the neutron spends within the reach of the nuclear force.
5. Assuming s wave interaction, we see that $\sigma_{\mathrm{e}}(e_{\mathrm{exc}})$ is almost constant at low energy and that the absorption-type reactions with no threshold energy vary as $1/\sqrt{E}$.