Ex 44 :

• Avec une paramétrisation trigonométrique :

$$\vec{R}(\theta,\varphi) = 3\cos\theta\sin\varphi\,\vec{i} + 3\sin\theta\sin\varphi\,\vec{j} + 3\cos\varphi\,\vec{k}$$

$$0 \leq \theta \leq 2\pi, \quad 0 \leq \varphi \leq \frac{\pi}{2}$$

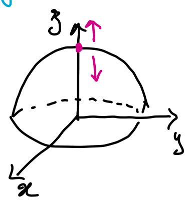
$$\vec{R_\theta}(\theta,\varphi) = -3\sin\theta\sin\varphi\,\vec{i} + 3\cos\theta\sin\varphi\,\vec{j} + 0\,\vec{k}$$

$$\vec{R_\varphi}(\theta,\varphi) = 3\cos\theta\cos\varphi\,\vec{i} + 3\sin\theta\cos\varphi\,\vec{j} - 3\sin\varphi\,\vec{k}$$

$$-9\cos\varphi\sin\varphi\,\vec{k}$$

$$\vec{R_\theta}(\theta,\varphi) \times \vec{R_\varphi}(\theta,\varphi) = -9\cos\theta\sin^2\varphi\,\vec{i} - (9\sin\theta\sin^2\varphi)\,\vec{j} + \boxed{(-9\sin^2\theta\cos\varphi\sin\varphi - 9\cos^2\theta\sin\varphi\cos\varphi)}\,\vec{k}$$

↳ vérification de l'orientation :



$\varphi = 0, \quad \theta \in [0, 2\pi]$

$$\|\vec{R_\theta} \times \vec{R_\varphi}\| = \sqrt{81\cos^2\theta\sin^4\varphi + 81\sin^2\theta\sin^4\varphi + 81\cos^2\varphi\sin^2\varphi}$$

$$= \sqrt{81\sin^4\varphi + 81\cos^2\varphi\sin^2\varphi}$$

$$= 9\sin\varphi\sqrt{\sin^2\varphi + \cos^2\varphi}$$

$$= 9\sin\varphi$$

$$\vec{n} = -\cos\theta\sin\varphi\,\vec{i} - \sin\theta\sin\varphi\,\vec{j} - \cos\varphi\,\vec{k}$$

$$\vec{n}(0,0) = 0\vec{i} + 0\vec{j} - \vec{k}$$

Ici le vecteur normal pointe vers l'intérieur. On va donc utiliser $-\vec{n}$, ie $\vec{R_\varphi} \times \vec{R_\theta}$

De plus, $\vec{F}(x,y,z) = \rho(x,y,z)\vec{v}(x,y,z)$

$\rho(x,y,z) = k$

$$= ky\,\vec{i} + kx\,\vec{j} + 0\,\vec{z}$$

donc $\vec{F}(\vec{R}(\theta,\varphi)) = k\sin\theta\sin\varphi\,\vec{i} + k\cos\theta\sin\varphi\,\vec{j} + 0\,\vec{j}$

$$\Phi = \iint_D k\sin\theta\sin\varphi \times 9\cos\theta\sin^2\varphi + k\cos\theta\sin\varphi \times 9\sin\theta\sin^2\varphi \, dA$$

$$= \iint_D 9k\cos\theta\sin\theta\sin^3\varphi + 9k\cos\theta\sin\theta\sin^3\varphi \, dA$$

$$= \iint_D 18k\cos\theta\sin\theta\sin^3\varphi \, dA$$

$$= \int_0^{2\pi}\int_0^{\pi/2} 18k\cos\theta\sin\theta\sin^3\varphi \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} 18k\cos\theta\sin\theta \times \frac{1}{12}\left(\cos(3\varphi) - 9\cos(\varphi)\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{3}{2}k\left(0 - 9\times 0 - 1 + 9\right)\int_0^{2\pi}\cos\theta\sin\theta \, d\theta$$

$$= 3k\frac{8}{4} \times 0$$

$$= 0$$

• Avec une paramétrisation cartesienne

$z = \sqrt{9 - x^2 - y^2}$

$\vec{R}(x,y) = x\vec{i} + y\vec{j} + \sqrt{9 - x^2 - y^2}\,\vec{k}$

$(x,y) \in D$, le cercle de rayon 3 centré en $(0,0)$

$\vec{R}_x(x,y) = \vec{i} + 0\vec{j} - \frac{x}{\sqrt{9-x^2-y^2}}\vec{k}$

$\vec{R}_y(x,y) = 0\vec{i} + \vec{j} - \frac{y}{\sqrt{9-x^2-y^2}}\vec{k}$

$\vec{R}_x \times \vec{R}_y = \frac{x}{\sqrt{9-x^2-y^2}}\vec{i} \quad \frac{y}{\sqrt{9-x^2-y^2}}\vec{j} + \vec{k}$

vérification de l'orientation : la composante en $\vec{k}$ est toujours ≥ 0 ✓

$\vec{F}(\vec{R}(x,y)) = \frac{k\,y}{\sqrt{9-x^2-y^2}}\vec{i} + \frac{k\,x}{\sqrt{9-x^2-y^2}}\vec{j} + 0\vec{k}$

$$\Phi \iint_D \frac{2kxy}{\sqrt{9-x^2-y^2}}\,dA = \int_0^{2\pi}\int_0^3 \frac{2kr^2\cos\theta\sin\theta}{\sqrt{9-r^2}}\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 2k\cos\theta\sin\theta\,\frac{r^3}{\sqrt{\underbrace{9-r^2}_{=u}}}\,dr\,d\theta$$

$u = 9 - r^2$

$\Rightarrow \frac{du}{dr} = -2r$

$\Rightarrow du = -2r\,dr$

$u = 9 - r^2 \Rightarrow r^2 = 9 - u$

$$= \int_0^{2\pi}\int_{u_1}^{u_2} -k\cos\theta\sin\theta\,\frac{\overbrace{r^2}^{=9-u}}{\sqrt{u}}\,dr\,d\theta$$

indépendant de θ

$$= \int_0^{2\pi} -k\cos\theta\sin\theta \boxed{\int_{u_1}^{u_2}\frac{9-u}{\sqrt{u}}\,du}\,d\theta$$

$$= -k\int_{u_1}^{u_2}\frac{9-u}{\sqrt{u}}\,du \boxed{\int_0^{2\pi}\cos\theta\sin\theta\,d\theta} = 0$$

$$= 0$$