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Simulation of multi-dimensional random fields by Karhunen–Loève expansion

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Highlights

- The paper presents a new method for simulating multi-dimensional random fields by Karhunen–Loève expansion.
- The proposed method is embedded into the well-established framework of K–L expansion for simulating stochastic process.
- Four examples illustrate the effectiveness of the proposed method.

Abstract

The Karhunen–Loève (K–L) expansion method is a powerful tool for simulating stationary and nonstationary, Gaussian and non-Gaussian stochastic processes with explicitly known covariance functions. Since the K–L expansion requires the solution of Fredholm integral equation of the second kind, it is generally not feasible to simulate multi-dimensional random fields. This is because even the numerical solution of the Fredholm integral multi-dimensional eigenvalue problem is difficult to obtain. In order to address this problem, this paper develops a consistent generalization of K–L expansion for multi-dimensional random field simulation. The new method decomposes an n-dimensional random field into a total of n stochastic processes, each can be represented by using the traditional K–L expansion. Thus, the developed method is embedded into the well-established framework of the K–L expansion for simulating stochastic process, and obviating the need for solving the multi-dimensional integral eigenvalue problems. Four examples, including random fields with different kinds of covariance functions, are used to demonstrate the application of the proposed method.

Keywords: Multi-dimensional random field; Karhunen–Loève expansion; Random field simulation; Fredholm integral equation

1. Introduction

In stochastic engineering problems, the proper consideration of the input variability is crucial to obtain an accurate and reliable solution. A large number of these problems involve uncertainty quantities which should be modeled as multi-dimensional random fields. The use of multi-dimensional random fields gained momentum due to the continued

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increase in available computational resources and nowadays is commonly used in many disciplines [1–3]. Several examples can be found in various fields of engineering. For instance, a number of common excitations, such as seismic ground motion, water velocity due to random waves, and wind velocity, can be reasonably modeled as random fields. For each of these environmental loads one often needs to use a multi-dimensional random field model to consider spatially correlated vector time histories of motion occurring simultaneously at different locations. In practice, it will often be the case that an amount of real realization of a random field are required. This is because the solution of practical engineering problems is often obtained through Monte Carlo simulation (MCS), in which the simulation of sample realizations of multi-dimensional random fields is one of the most important stages [4–6]. On the other hand, in the context of stochastic finite element analysis of structures, the discretization of continuous random fields becomes necessary. For computational purposes, the discretization must represent the continuous random field with sufficient accuracy with as few random variables as possible [7–9]. As a consequence, the simulation and discretization of multi-dimensional random fields is of practical and theoretical importance.

According to [10], the two most widely used approaches for simulating one-dimensional random field, known as stochastic process, are based on the spectral representation (discretization in the frequency domain) or on the Karhunen-Loève (K–L) expansion (discretization in the physical domain, e.g., time or space). Although the former has been successfully extended for simulating multi-dimensional random fields, e.g.,[11], the present paper is focused on method that based on K–L expansion since it is a powerful tool for representing stationary and nonstationary, Gaussian and non-Gaussian stochastic process with explicitly known covariance functions [12–14]. Since K–L expansion is optimal among series expansion methods in the global mean square error with respect to the number of random variables in the representation, it has received much attention in many disciplines. For instance, the K–L expansion (also known as proper orthogonal decomposition in most of the turbulence literatures) has become a standard tool for extracting the most energetic modes from a fluctuating velocity field that define the coherent structure of the fluid flow [15]. In structural dynamics and random vibration, it has been applied to identify the modal shapes of two- or three-dimensional structures having nonhomogeneous density [16,17] and to determine the stochastic response and reliability of nonlinear system [18–20]. In optimization problems, the K–L expansion has been successfully applied to assess the input variability associated to design spaces in shape optimization and build a reduced-dimensionality representation of the shape modification vector [21,22]. In stochastic finite element analysis, it has been widely used to discretize the random fields representing the randomness of structures and excitation [8,10]. It is worth mentioning that the implementation of K–L expansion requires the solution of a Fredholm integral equation of the second kind with the covariance function of the stochastic process as the integral kernel. Although only a limited number of analytical eigen-solutions are available, the solution of the integral equation can be numerically approximated for stochastic processes with arbitrary covariance functions [23–25]. However, for the multi-dimensional random fields, it is generally not feasible to obtain the numerical solution of the Fredholm integral multi-dimensional eigenvalue problem [26–28]. This is why, in the last few years, most applications of K–L expansion only focus on simulating stochastic process. For random fields that are defined on two- and three-dimensional domains, the finite element method becomes the only available method for the discretization of the multi-dimensional integral eigenvalue problems [10,29]. However, on two- and especially three-dimensional random field domains, the generation of a finite element mesh is an involved task and the computational costs become expensive [7].

Borrow the idea that proposed by the present authors for handling the multi-dimensional moment-constrained maximum entropy problem [30], this paper develops a consistent generalization of K–L expansion for multi-dimensional random fields simulation. The new method expands an n-dimensional random field into a one-dimensional stochastic process and an $(n-1)$-dimensional random field, the obtained $(n-1)$-dimensional random field is further decomposed into a new one-dimensional stochastic process and an $(n-2)$-dimensional random field. By repeating the above process, the original n-dimensional random field is decomposed into a series of one-dimensional stochastic processes step by step, each can be represented by using traditional K–L expansion. Thus, the developed method is embedded into the well-established framework of the K–L expansion for simulating stochastic process, and obviating the need for the finite element method for solving the multi-dimensional integral eigenvalue problems.

The paper is organized as follows: the traditional K–L expansion is briefly introduced in Section 2, followed by the developed method for simulating multi-dimensional random fields. Four illustrative examples are finally given in Section 4 to demonstrate the application of the proposed method.

2. K–L expansion of one-dimensional random process

The Karhunen-Loève expansion is a series expansion method for the representation of the random process. The expansion is based on a spectral decomposition of the covariance function of the process. It states that a second-order random process $w(t,\theta)$, which is indexed on a bounded domain $\mathcal{D}$, can be represented exactly by the following expansion

$$w(t,\theta) = \bar{w}(t) + \sum_{n=1}^{\infty} \sqrt{\lambda_n}\xi_n(\theta) f_n(t) \tag{1}$$

where $\bar{w}(t)$ is the mean function of the process, $\xi_n(\theta)$ are standard normal uncorrelated random variables, and λ_n and $f_n(t)$ are the eigenvalues and eigenfunctions of the covariance function $C(t_1,t_2)$ of the process, obtained from solving the following homogeneous Fredholm integral equation of the second kind:

$$\int_{\mathcal{D}} C(t_1,t_2) f_n(t_1) dt_1 = \lambda_n f_n(t_2). \tag{2}$$

Since the covariance function is bounded, symmetric and positive definite, $C(t_1,t_2)$ has the spectral decomposition as

$$C(t_1,t_2) = \sum_{n=1}^{\infty} \lambda_n f_n(t_1) f_n(t_2). \tag{3}$$

According to Mercer's theorem, the eigenvalues λ_n are nonnegative, the eigenfunctions corresponding to positive eigenvalues are continuous and orthogonal to each other, i.e., $\int_{\mathcal{D}} f_n(t) f_m(t) dt = \delta_{nm}$, where δ_{nm} is one if $n = m$ and zero otherwise. The random variables $\xi_n(\theta)$ in Eq. (1) are expressed as

$$\xi_n(\theta) = \frac{1}{\sqrt{\lambda_n}} \int_{\mathcal{D}} [w(t,\theta) - \bar{w}(t)]\, f_n(t)\, dt \tag{4}$$

with mean and covariance function given by

$$E[\xi_n(\theta)] = 0, \;\; E[\xi_n(\theta)\xi_m(\theta)] = \delta_{nm}. \tag{5}$$

Obviously, the use of K–L expansion with orthogonal deterministic basis functions and uncorrelated random coefficients has generated interest because of its bi-orthogonal property. This allows for the optimal encapsulation of the information contained in the random process into a set of discrete uncorrelated random variables [31,32]. For practical implementation, the K–L expansion can be approximated by sorting the eigenvalues λ_n and the corresponding eigenfunctions $f_n(t)$ in a descending order and truncating the expansion after M terms:

$$\hat{w}(t,\theta) = \bar{w}(t) + \sum_{n=1}^{M} \sqrt{\lambda_n}\xi_n(\theta) f_n(t). \tag{6}$$

For fixed M, the resulting random process approximation $\hat{w}(t,\theta)$ is optimal among series expansion methods with respect to the global mean square error [10]. The covariance of $\hat{w}(t,\theta)$ is given by

$$\hat{C}(t_1,t_2) = \sum_{n=1}^{M} \lambda_n f_n(t_1) f_n(t_2). \tag{7}$$

3. Simulation of multi-dimensional random fields by K–L expansion

It is known that the application of K–L expansion for simulating random fields hinges crucially on the ability to solve the integral eigenvalue problems of the type given in Eq. (2). Analytical solutions can be obtained only for specific types of covariance functions. For random fields with arbitrary covariance functions defined on domains of complex geometrical shape, the solution of the integral eigenvalue problems needs to be approximated numerically. For the covariance functions defined on multi-dimensional domain, one has to solve the corresponding multi-dimensional Fredholm integral equation, which is a challenging problem. In order to avoid numerically solving the multi-dimensional integral eigenvalue problem in the simulation of a multi-dimensional random field, we develop a new method on the basis of traditional K–L expansion. We describe the new method by using a 2-dimensional

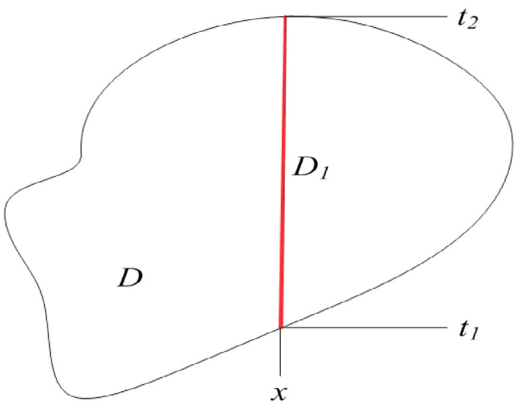


Fig. 1. Relationship between $\mathcal{D}_1(x)$ and $\mathcal{D}$.

random field in Section 3.1, and then generalize the method for simulating an arbitrary n-dimensional random field with explicitly known covariance function in Section 3.2. The efficiency of the proposed method is finally discussed in Section 3.3.

3.1. Simulation of a two-dimensional random field

Consider a two-dimensional random field $w(x, t, \theta)$ that defined on a probability space (Ω, A, P) and indexed on a bounded domain $\mathcal{D}$. Without loss of generality, we assume that the field has a zero mean and a finite covariance function $C(x_1, x_2, t_1, t_2)$, which is bounded for all $x, t \in \mathcal{D}$. Fix the position vector, i.e., $x_1 = x_2 = x$, then the covariance function $C(x_1, x_2, t_1, t_2)$ becomes a 'quasi' one-dimensional function $C(x, t_1, t_2)$, and in this case the random field $w(x, t, \theta)$ can be expanded as

$$w(x, t, \theta) = \sum_{n=1}^{\infty} \xi_n(x, \theta)\sqrt{\lambda_n(x)} f_n(t, x) \tag{8}$$

where $\lambda_n(x)$ and $f_n(t, x)$ are the eigenvalues and eigenfunctions of the covariance function $C(x, t_1, t_2)$, respectively. Note that both $\lambda_n(x)$ and $f_n(t, x)$ are functions of the position variable x. By definition, $C(x, t_1, t_2)$ is bounded, symmetric and positive definite. Therefore, it has the spectral decomposition as

$$C(x, t_1, t_2) = \sum_{n=1}^{\infty} \lambda_n(x) f_n(t_1, x) f_n(t_2, x). \tag{9}$$

The eigenvalues and eigenfunctions of $C(x, t_1, t_2)$ are solutions of the following Fredholm integral equation:

$$\int_{\mathcal{D}_1(x)} C(x, t_1, t_2) f_n(t_1, x) dt_1 = \lambda_n(x) f_n(t_2, x) \tag{10}$$

where $\mathcal{D}_1(x)$, the sub-domain of $\mathcal{D}$, is dependent on the position variable x. Fig. 1 geometrically describes the relations between $\mathcal{D}_1(x)$ and $\mathcal{D}$. The eigenfunctions are orthogonal and form a complete set satisfying

$$\int_{\mathcal{D}_1(x)} f_n(t, x) f_m(t, x) dt = \delta_{nm}. \tag{11}$$

It can be seen that the parameter $\xi_n(x, \theta)$ in Eq. (8) is the function of position variable x and hence can be considered as a stochastic process. Thus, $\xi_n(x, \theta)$ can be expressed as

$$\xi_n(x, \theta) = \frac{1}{\sqrt{\lambda_n(x)}} \int_{\mathcal{D}_1(x)} w(x, t, \theta) f_n(t, x) dt \tag{12}$$

with mean and covariance function given by

$$E\left[\xi_n(x, \theta)\right] = 0, \quad E\left[\xi_n(x, \theta)\xi_m(x, \theta)\right] = \delta_{nm}. \tag{13}$$

Obviously, as long as this intermediate one-dimensional stochastic process $\xi_n(x,\theta)$ is simulated, the original two-dimensional random field $w(x,t,\theta)$ can be represented just by using the expansion in Eq. (8). Hence, it is required to explicitly derive the covariance function of $\xi_n(x,\theta)$ so that this process can be simulated by means of K–L expansion. To implement this, we denote $H_n(x_1,x_2)$ as the covariance function of $\xi_n(x,\theta)$. By substituting Eq. (12) into the definition of a covariance function, $H_n(x_1,x_2)$ is derived as

$$\begin{aligned} H_n(x_1,x_2) &= E\left[\xi_n(x_1,\theta)\xi_n(x_2,\theta)\right] \\ &= E\left[\frac{1}{\sqrt{\lambda_n(x_1)}}\int_{\mathcal{D}_1(x_1)} w(x_1,t_1,\theta) f_n(t_1,x_1)\,dt_1 \frac{1}{\sqrt{\lambda_n(x_2)}}\int_{\mathcal{D}_1(x_2)} w(x_2,t_2,\theta) f_n(t_2,x_2)\,dt_2\right] \\ &= \frac{1}{\sqrt{\lambda_n(x_1)\lambda_n(x_2)}}\iint_{\mathcal{D}_1(x_1)\times\mathcal{D}_1(x_2)} E\left[w(x_1,t_1,\theta)w(x_2,t_2,\theta)\right] f_n(t_1,x_1)\, f_n(t_2,x_2)\,dt_1dt_2 \\ &= \frac{1}{\sqrt{\lambda_n(x_1)\lambda_n(x_2)}}\iint_{\mathcal{D}_1(x_1)\times\mathcal{D}_1(x_2)} C(x_1,x_2,t_1,t_2) f_n(t_1,x_1)\, f_n(t_2,x_2)\,dt_1dt_2 \end{aligned} \tag{14}$$

where $E[\cdot]$ denotes the mathematical expectation. Thus, this intermediate stochastic process $\xi_n(x,\theta)$ in Eq. (8) can be simulated by using the K–L expansion as

$$\xi_n(x,\theta) = \sum_{k=1}^{\infty} \eta_{nk}(\theta)\sqrt{\mu_{nk}}\, g_{nk}(x) \tag{15}$$

where μ_{nk} and $g_{nk}(x)$ are eigenvalues and eigenfunctions of the covariance function $H_n(x_1,x_2)$, a solution of the integral eigenvalue problem:

$$\int_{\mathcal{D}_1} H_n(x_1,x_2) g_{nk}(x_2)\,dx_2 = \mu_{nk} g_{nk}(x_1) \tag{16}$$

and $\eta_{nk}(\theta)$ is a set of uncorrelated random variables satisfying $E[\eta_{nk}(\theta)] = 0$ and $E[\eta_{nk}(\theta)\eta_{nl}(\theta)] = \delta_{kl}$.

Thus, by substituting the representation of $\xi_n(x,\theta)$ in Eq. (15) into the K–L expansion as given in Eq. (8), the original two-dimensional random field $w(x,t,\theta)$ is expanded as

$$w(x,t,\theta) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \eta_{nk}(\theta)\sqrt{\lambda_n(x)}\, f_n(t,x)\sqrt{\mu_{nk}}\, g_{nk}(x). \tag{17}$$

The covariance function of the field $w(x,t,\theta)$ has the following spectral decomposition:

$$C(x_1,x_2,t_1,t_2) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \sqrt{\lambda_n(x_1)\lambda_n(x_2)}\, f_n(t_1,x_1) f_n(t_2,x_2)\mu_{nk} g_{nk}(x_1) g_{nk}(x_2). \tag{18}$$

As a summary, the two-dimensional random field $w(x,t,\theta)$ is firstly decomposed into a set of deterministic functions and corresponding one-dimensional random processes, and these obtained intermediate random processes are then expanded by K–L expansion one more time. In this way, the original two-dimensional random field is represented just by using traditional K–L expansion twice.

For the autocorrelation function of $\xi_n(x,\theta)$, since the position vector is fixed as $x_1 = x_2 = x$, $\xi_n(x_1,\theta)$ and $\xi_n(x_2,\theta)$ can be considered as random variables. Thus, $H_n(x,x)$ can be derived as

$$\begin{aligned} H_n(x,x) &= E\left[\xi_n(x,\theta)\xi_n(x,\theta)\right] \\ &= \frac{1}{\lambda_n(x)}\int\int_{\mathcal{D}_1(x)\times\mathcal{D}_1(x)} C(x,t_1,t_2) f_n(t_1,x) f_n(t_2,x) dt_1dt_2 \\ &= \frac{1}{\lambda_n(x)}\int_{\mathcal{D}_1(x)}\left[\int_{\mathcal{D}_1(x)} C(x,t_1,t_2) f_n(t_1,x) dt_1\right] f_n(t_2,x) dt_2. \end{aligned} \tag{19}$$

By substituting Eq. (10) into Eq. (19), $H_n(x,x)$ can be further simplified as

$$H_n(x,x) = E\left[\xi_n(x,\theta)\xi_n(x,\theta)\right] = \int_{\mathcal{D}_1(x)} f_n(t_2,x)\, f_n(t_2,x)\,dt_2 = 1. \tag{20}$$

Obviously, Eq. (20) coincides with the property that given in Eq. (13). In addition, since the cross-correlation function of $\xi_n(x,\theta)$ and $\xi_m(x,\theta)$ can be derived as

$$\begin{aligned} E\left[\xi_n(x,\theta)\xi_m(x,\theta)\right] &= E\left[\sum_{k=1}^{\infty}\eta_{nk}(\theta)\sqrt{\mu_{nk}}g_{nk}(x)\sum_{l=1}^{\infty}\eta_{ml}(\theta)\sqrt{\mu_{ml}}g_{ml}(x)\right] \\ &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}E\left[\eta_{nk}(\theta)\eta_{ml}(\theta)\right]\sqrt{\mu_{nk}\mu_{ml}}g_{nk}(x)g_{ml}(x). \end{aligned} \tag{21}$$

If the condition

$$E\left[\eta_{nk}(\theta)\eta_{ml}(\theta)\right] = \delta_{nm}\delta_{kl} \tag{22}$$

is satisfied, then Eq. (21) reduces to

$$\begin{aligned} E\left[\xi_n(x,\theta)\xi_m(x,\theta)\right] &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}E\left[\eta_{nk}(\theta)\eta_{ml}(\theta)\right]\sqrt{\mu_{nk}\mu_{ml}}g_{nk}(x)g_{ml}(x) \\ &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\delta_{nm}\delta_{kl}\sqrt{\mu_{nk}\mu_{ml}}g_{nk}(x)g_{ml}(x) \\ &= \delta_{nm}\sum_{k=1}^{\infty}\mu_{nk}g_{nk}^2(x) = \delta_{nm}H_n(x,x) = \delta_{nm}. \end{aligned} \tag{23}$$

Thus, Eq. (23) also coincides with the property that given in Eq. (13).

3.2. Simulation of a multi-dimensional random field

The expansion that proposed in Section 3.1 can be straightforwardly extended to three- or multi-dimensional random fields simulation. For a zero-mean three-dimensional random field $w(x,y,t,\theta)$, following the way for expanding a two-dimensional random field in Section 3.1, we firstly fix the position vector $x_1 = x_2 = x$ and $y_1 = y_2 = y$, and determine the eigenvalues and eigenfunctions of the 'quasi' two-dimensional covariance $C(x,y,t_1,t_2)$, the corresponding random coefficients would be a set of intermediate two-dimensional random fields. Since these intermediate random fields can be further expanded by using the method that given in Eq. (17), the original three-dimensional random field $w(x,y,t,\theta)$ can be finally represented by

$$w(x,y,t,\theta) = \sum_{i=1}^{\infty}\sqrt{\lambda_i(x,y)}f_i(t,x,y)\sum_{j=1}^{\infty}\sqrt{\mu_{ij}(y)}g_{ij}(x,y)\sum_{k=1}^{\infty}\eta_{ijk}(\theta)\sqrt{\nu_{ijk}}p_{ijk}(y). \tag{24}$$

In order to concisely generalize Eq. (24) to a multi-dimensional case, it can be rewritten in a more unified form as

$$w(x_1,x_2,x_3,\theta) = \sum_{i_1=1}^{\infty}\sqrt{\lambda_{i_1}(x_2,x_3)}f_{i_1}(x_1,x_2,x_3)\sum_{i_2=1}^{\infty}\sqrt{\lambda_{i_1i_2}(x_3)}f_{i_1i_2}(x_2,x_3)\sum_{i_3=1}^{\infty}\eta_{i_1i_2i_3}(\theta)\sqrt{\lambda_{i_1i_2i_3}}f_{i_1i_2i_3}(x_3). \tag{25}$$

Thus, the notations in Eq. (25) can be easily extended to represent an arbitrary n-dimensional random field $w(x_1,\ldots,x_n;\theta)$ as:

$$\begin{aligned} w(x_1,\ldots,x_n;\theta) &= \sum_{i_1=1}^{\infty}\sqrt{\lambda_{i_1}(x_2,\ldots,x_n)}f_{i_1}(x_1,\ldots,x_n)\sum_{i_2=1}^{\infty}\sqrt{\lambda_{i_1i_2}(x_3,\ldots,x_n)}f_{i_1i_2}(x_2,\ldots,x_n) \\ &\cdots\sum_{i_{n-1}=1}^{\infty}\sqrt{\lambda_{i_1\cdots i_{n-1}}(x_n)}f_{i_1\cdots i_{n-1}}(x_{n-1},x_n)\sum_{i_{n-1}=1}^{\infty}\eta_{i_1,\ldots,i_n}(\theta)\sqrt{\lambda_{i_1\cdots i_n}}f_{i_1\cdots i_n}(x_n) \end{aligned} \tag{26}$$

where subscript n represents the nth dimensionality. The covariance function of $w(x_1,\ldots,x_n;\theta)$ has the following spectral decomposition

$$\begin{aligned} &C(x_{11},x_{12},x_{21},x_{22},\ldots,x_{n1},x_{n2}) \\ &= \sum_{i_1=1}^{\infty}\sqrt{\lambda_{i_1}(x_{21},\ldots,x_{n1})\lambda_{i_1}(x_{22},\cdot\ldots\cdot,x_{n2})}f_{i_1}(x_{11},\ldots,x_{n1})f_{i_1}(x_{12},\ldots,x_{n2}) \end{aligned}$$

$$\times \sum_{i_2=1}^{\infty} \sqrt{\lambda_{i_1 i_2}(x_{31},\ldots,x_{n1})\lambda_{i_1 i_2}(x_{32},\ldots,x_{n2})}\, f_{i_1 i_2}(x_{21},\ldots,x_{n1}) f_{i_1 i_2}(x_{22},\ldots,x_{n2})$$
$$\times \cdots \sum_{i_{n-1}=1}^{\infty} \sqrt{\lambda_{i_1\cdots i_{n-1}}(x_{n1})\lambda_{i_1\cdots i_{n-1}}(x_{n2})}\, f_{i_1\cdots i_{n-1}}(x_{n-1,1}, x_{n1}) f_{i_1\cdots i_{n-1}}(x_{n-1,2}, x_{n2})$$
$$\times \sum_{i_{n-1}=1}^{\infty} \lambda_{i_1\cdots i_n} f_{i_1\cdots i_n}(x_{n1}) f_{i_1\cdots i_n}(x_{n2}). \tag{27}$$

In this way, an n-dimensional random field is expanded to a total of n one-dimensional stochastic processes step by step, each can be represented by using traditional K–L expansion. Note that the limits of each summation in Eqs. (26) and (27) are from one to infinity. Since each summation means one round of K–L expansion for simulating the corresponding one-dimensional stochastic process, the series in each summation can be approximated by a finite number of terms, just like the cases in Eqs. (6) and (7) as

$$\hat{w}(x_1,\ldots,x_n;\theta) = \sum_{i_1=1}^{M_1} \sqrt{\lambda_{i_1}(x_2,\ldots,x_n)}\, f_{i_1}(x_1,\ldots,x_n) \sum_{i_2=1}^{M_2} \sqrt{\lambda_{i_1 i_2}(x_3,\ldots,x_n)}\, f_{i_1 i_2}(x_2,\ldots,x_n)$$
$$\cdots \sum_{i_{n-1}=1}^{M_{n-1}} \sqrt{\lambda_{i_1\cdots i_{n-1}}(x_n)}\, f_{i_1\cdots i_{n-1}}(x_{n-1}, x_n) \sum_{i_n=1}^{M_n} \eta_{i_1,\ldots,i_n}(\theta)\sqrt{\lambda_{i_1\cdots i_n}}\, f_{i_1\cdots i_n}(x_n) \tag{28}$$

and the corresponding covariance function is approximated as

$$\begin{aligned}
&\hat{C}(x_{11}, x_{12}, x_{21}, x_{22}, \ldots, x_{n1}, x_{n2}) \\
&= \sum_{i_1=1}^{M_1} \sqrt{\lambda_{i_1}(x_{21},\cdot\ldots\cdot,x_{n1})\lambda_{i_1}(x_{22},\cdot\ldots\cdot,x_{n2})}\, f_{i_1}(x_{11},\ldots,x_{n1}) f_{i_1}(x_{12},\ldots,x_{n2}) \\
&\quad \times \sum_{i_2=1}^{M_2} \sqrt{\lambda_{i_1 i_2}(x_{31},\ldots,x_{n1})\lambda_{i_1 i_2}(x_{32},\ldots,x_{n2})}\, f_{i_1 i_2}(x_{21},\ldots,x_{n1}) f_{i_1 i_2}(x_{22},\ldots,x_{n2}) \\
&\quad \cdots \times \sum_{i_{n-1}=1}^{M_{n-1}} \sqrt{\lambda_{i_1\cdots i_{n-1}}(x_{n1})\lambda_{i_1\cdots i_{n-1}}(x_{n2})}\, f_{i_1\cdots i_{n-1}}(x_{n-1,1}, x_{n1}) f_{i_1\cdots i_{n-1}}(x_{n-1,2}, x_{n2}) \\
&\quad \times \sum_{i_{n-1}=1}^{M_n} \lambda_{i_1\cdots i_n} f_{i_1\cdots i_n}(x_{n1}) f_{i_1\cdots i_n}(x_{n2}).
\end{aligned} \tag{29}$$

For sufficiently large $M_i, i = 1, 2, \ldots, N$ in each summation, the second-moment properties of the set of intermediate stochastic processes can be approximated by the second-moment properties of the corresponding K–L representation, and the accuracy of each round of K–L approximation depends on the value of M_i. As has been proved in [10], the mean square error resulting from the K–L truncation is minimized, the truncated series in each summation is thus converge to the corresponding one-dimensional stochastic process with optimal representation. Once the intermediate process of the first dimension has been represented by the corresponding truncated K–L representation, the K–L approximation of the next-dimensional intermediate process will also converge with minimum mean square error. In this sense, the K–L expansion in Eq. (28) is optimal with respect to the mean square approximation error. Take the representation of the two-dimensional random field in Eq. (17) as an example, once the intermediate stochastic process $\xi_n(x, \theta)$ can be represented by truncating the K–L expansion in Eq. (15) with $k = M_2$ terms, the summation of finite series in Eq. (8) (i.e., $n = M_1$) can be used to approximate the original random field $w(x, t, \theta)$ with minimum mean square approximation error.

3.3. Computation of the covariance of intermediate stochastic process

It has been shown that a total of n intermediate one-dimensional stochastic processes are produced during the process for simulating an n-dimensional random field. Obviously, Eq. (26) or Eq. (28) can be employed only when all of these one-dimensional stochastic processes are simulated. In this section, we will take the two-dimensional random field $w(x, t, \theta)$ as an example to develop an effective method for simulating the intermediate one-dimensional stochastic processes and then generalize the method to multi-dimensional case.

Recall that the covariance function $H_n(x_1, x_2)$ of the intermediate one-dimensional stochastic process $\xi_n(x, \theta)$ in Eq. (14) involves the term $\frac{1}{\sqrt{\lambda_n(x_1)\lambda_n(x_2)}}$, which are functions of variables x_1 and x_2. This may lead to ill-posed condition in the computation of $H_n(x_1, x_2)$ because $\sqrt{\lambda_n(x_1)}$ and $\sqrt{\lambda_n(x_2)}$ are in the denominator of $H_n(x_1, x_2)$. In order to avoid this problem, we consider a new one-dimensional stochastic process $\sqrt{\lambda_n(x)}\xi_n(x, \theta)$ instead of $\xi_n(x, \theta)$. From Eq. (14), the covariance of the new process $\sqrt{\lambda_n(x)}\xi_n(x, \theta)$ can be easily derived as

$$\begin{aligned} R_n(x_1, x_2) &= E\left[\sqrt{\lambda_n(x_1)}\xi_n(x_1, \theta)\sqrt{\lambda_n(x_2)}\xi_n(x_2, \theta)\right] \\ &= \iint_{\mathcal{D}_1(x_1)\times\mathcal{D}_1(x_2)} C(x_1, x_2, t_1, t_2) f_n(t_1, x_1) f_n(t_2, x_2) dt_1 dt_2 \end{aligned} \tag{30}$$

where $R_n(x_1, x_2)$ denotes the covariance function of the process $\sqrt{\lambda_n(x)}\xi_n(x, \theta)$. Compared $R_n(x_1, x_2)$ with $H_n(x_1, x_2)$, it is not difficult to find that the only difference between these two covariance kernels is that the term $\frac{1}{\sqrt{\lambda_n(x_1)\lambda_n(x_2)}}$ in $H_n(x_1, x_2)$ is diminished. Once the covariance $R_n(x_1, x_2)$ is determined, the stochastic process $\sqrt{\lambda_n(x)}\xi_n(x, \theta)$ can be simulated by using K–L expansion as

$$\sqrt{\lambda_n(x)}\xi_n(x, \theta) = \sum_{k=1}^{\infty} \tilde{\eta}_{nk}(\theta)\sqrt{\tilde{\mu}_{nk}}\tilde{g}_{nk}(x) \tag{31}$$

where $\tilde{\mu}_{nk}$ and $\tilde{g}_{nk}(x)$ are the eigenvalues and eigenfunctions of the covariance $R_n(x_1, x_2)$, satisfying

$$\int_{\mathcal{D}_1} R_n(x_1, x_2)\tilde{g}_{nk}(x_2) dx_2 = \tilde{\mu}_{nk}\tilde{g}_{nk}(x_1) \tag{32}$$

and $\tilde{\eta}_{nk}(\theta)$ is the corresponding set of uncorrelated random coefficients. Thus, by substituting Eq. (31) into Eq. (8), the K–L expansion of the two-dimensional random field $w(x, t, \theta)$ in Eq. (17) can be rewritten as

$$w(x, t, \theta) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \tilde{\eta}_{nk}(\theta) f_n(t, x)\sqrt{\tilde{\mu}_{nk}}\tilde{g}_{nk}(x) \tag{33}$$

and the covariance function in Eq. (18) is accordingly simplified as

$$C(x_1, x_2, t_1, t_2) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_n(t_1, x_1) f_n(t_2, x_2)\tilde{\mu}_{nk}\tilde{g}_{nk}(x_1)\tilde{g}_{nk}(x_2). \tag{34}$$

Obviously, due to the simplified form of the covariance kernel, the Fredholm integral equation in Eq. (32) is much easier to solve than that in Eq. (16). As a result, the expansion in Eq. (33) is more effective than that in Eq. (17) for two-dimensional random field simulation.

The above method for efficiently computing the covariance function of the intermediate one-dimensional stochastic process, and the subsequent Fredholm integral equation, can be straightforwardly extended to the multi-dimensional random field simulation. For an arbitrary n-dimensional random field $w(x_1, \ldots, x_n; \theta)$, the expansion in Eq. (26) can be rewritten as

$$\begin{aligned} w(x_1, \ldots, x_n; \theta) = &\sum_{i_1=1}^{\infty} f_{i_1}(x_1, \ldots, x_n) \sum_{i_2=1}^{\infty} f_{i_1 i_2}(x_2, \ldots, x_n) \\ &\cdots \sum_{i_{n-1}=1}^{\infty} f_{i_1\cdots i_{n-1}}(x_{n-1}, x_n) \sum_{i_{n-1}=1}^{\infty} \eta_{i_1,\ldots,i_n}(\theta)\sqrt{\lambda_{i_1\cdots i_n}} f_{i_1\cdots i_n}(x_n) \end{aligned} \tag{35}$$

with the covariance function simplified by

$$\begin{aligned} &C(x_{11}, x_{12}, x_{21}, x_{22}, \ldots, x_{n1}, x_{n2}) \\ &= \sum_{i_1=1}^{\infty} f_{i_1}(x_{11}, \ldots, x_{n1}) f_{i_1}(x_{12}, \ldots, x_{n2}) \sum_{i_2=1}^{\infty} f_{i_1 i_2}(x_{21}, \ldots, x_{n1}) f_{i_1 i_2}(x_{22}, \ldots, x_{n2}) \\ &\quad \times \cdots \sum_{i_{n-1}=1}^{\infty} f_{i_1\cdots i_{n-1}}(x_{n-1,1}, x_{n1}) f_{i_1\cdots i_{n-1}}(x_{n-1,2}, x_{n2}) \sum_{i_{n-1}=1}^{\infty} \lambda_{i_1\cdots i_n} f_{i_1\cdots i_n}(x_{n1}) f_{i_1\cdots i_n}(x_{n2}). \end{aligned} \tag{36}$$

By using the truncated K–L expansion in Eqs. (28) and (29), Eqs. (35) and (36) can be, respectively, approximated by

$$\begin{aligned}\hat{w}(x_1,\ldots,x_n;\theta) = &\sum_{i_1=1}^{M_1} f_{i_1}(x_1,\ldots,x_n)\sum_{i_2=1}^{M_2} f_{i_1 i_2}(x_2,\ldots,x_n)\\ &\cdots\sum_{i_{n-1}=1}^{M_{n-1}} f_{i_1\cdots i_{n-1}}(x_{n-1},x_n)\sum_{i_n=1}^{M_n}\eta_{i_1,\ldots,i_n}(\theta)\sqrt{\lambda_{i_1\cdots i_n}}f_{i_1\cdots i_n}(x_n)\end{aligned} \tag{37}$$

and

$$\begin{aligned}&\hat{C}(x_{11},x_{12},x_{21},x_{22},\ldots,x_{n1},x_{n2})\\ &=\sum_{i_1=1}^{M_1} f_{i_1}(x_{11},\ldots,x_{n1})f_{i_1}(x_{12},\ldots,x_{n2})\sum_{i_2=1}^{M_2} f_{i_1 i_2}(x_{21},\ldots,x_{n1})f_{i_1 i_2}(x_{22},\ldots,x_{n2})\\ &\times\cdots\sum_{i_{n-1}=1}^{M_{n-1}} f_{i_1\cdots i_{n-1}}(x_{n-1,1},x_{n1})f_{i_1\cdots i_{n-1}}(x_{n-1,2},x_{n2})\sum_{i_{n-1}=1}^{M_n}\lambda_{i_1\cdots i_n}f_{i_1\cdots i_n}(x_{n1})f_{i_1\cdots i_n}(x_{n2}).\end{aligned} \tag{38}$$

Specifically, in the simulation of a two-dimensional random field $w(x,t,\theta)$, Eqs. (33) and (34) can be accordingly approximated by

$$\hat{w}(x,t,\theta)=\sum_{n=1}^{M_1} f_n(t,x)\sum_{k=1}^{M_2}\tilde{\eta}_{nk}(\theta)\sqrt{\tilde{\mu}_{nk}}\tilde{g}_{nk}(x) \tag{39}$$

and

$$\hat{C}(x_1,x_2,t_1,t_2)=\sum_{n=1}^{M_1}\sum_{k=1}^{M_2} f_n(t_1,x_1)f_n(t_2,x_2)\tilde{\mu}_{nk}\tilde{g}_{nk}(x_1)\tilde{g}_{nk}(x_2). \tag{40}$$

Compared with Eq. (28), Eq. (37) is more applicable for simulating an n-dimensional random field not only due to its simplified form, the more important reason is that the Fredholm integral equation with the covariance kernel $R_n(x_1,x_2)$ is more easy to solve than that with the kernel $H_n(x_1,x_2)$. Therefore, Eq. (37) is proposed as a general formula to represent multi-dimensional random field in practice. The general procedure of the proposed method for simulating an n-dimensional random field is firstly to fix the position vector to determine the eigenvalues and eigenfunctions of the corresponding quasi covariance function, and then to construct the covariance function $R_n(x_1,x_2)$ of the intermediate $(n-1)$-dimensional random field according to Eq. (30), and finally solve the integral eigenvalues problem that associated with the kernel $R_n(x_1,x_2)$. When the above process is repeated by n times, the original n-dimensional random field is decomposed into a total of n one-dimensional stochastic processes, each can be represented by using the traditional K–L expansion. In this way, the proposed method is embedded into the well-established framework of the K–L expansion.

4. Numerical examples

In this section, four illustrative examples are provided to examine the effectiveness of the proposed method. The first two examples consider a stationary and a non-stationary two-dimensional random field with separable covariance function. Note that in such cases, the traditional K–L is also available for simulating the random field by assuming that the eigenfunctions possess the separate structure. In order to investigate the performance of the proposed method for simulating a more general random field, examples 3 and 4 consider the two-dimensional random field with non-separable covariance. In all examples, the results of the proposed method are compared with the exact solutions.

4.1. Example 1: Stationary random field with separable covariance

The first example considers a zero-mean two-dimensional random field $w(x,t,\theta)$ with covariance function given by

$$C(x_1,x_2,t_1,t_2)=\left(1-d\left|x_1-t_1\right|\right)\left(1-d\left|x_2-t_2\right|\right),\ \left|x_1-t_1\right|\in\left[0,\frac{1}{d}\right],\ \left|x_2-t_2\right|\in\left[0,\frac{1}{d}\right] \tag{41}$$

where d is a parameter that is used to adjust the distance $|x_i - t_i|$, $i = 1, 2$ of null correlation between $w(x_1, t_1, \theta)$ and $w(x_2, t_2, \theta)$. In this example, d is adopted as one. To implement the proposed method to simulate $w(x, t, \theta)$, we firstly fix $x_1 = t_1 = x$ to obtain a 'quasi' one-dimensional covariance as

$$C(x, x_2, t_2) = 1 - |x_2 - t_2|, \ |x_2 - t_2| \in [0, 1] \tag{42}$$

and determine the eigenfunctions of the covariance $C(x, x_2, t_2)$. Consider realizations of the corresponding process on interval $[0, 1]$, then according to Eq. (10), the homogeneous Fredholm integral equation of kernel $C(x, x_2, t_2)$ becomes

$$\int_0^1 [1 - |x_2 - t_2|] f_i(x_2, x) dx_2 = \lambda_i(x) f_i(t_2, x). \tag{43}$$

The closed-form solution of Eq. (43) can be found in [10] as

$$f_i(t, x) = \frac{\cos \omega_i t}{\sqrt{\frac{1}{2} + \frac{\sin 2\omega_i}{2\omega_i}}} \tag{44}$$

where

$$\omega_i = \begin{cases} i\pi, & i = 2k - 1 \\ \dfrac{2}{\tan \omega_i / 2}, & i = 2k. \end{cases} \tag{45}$$

Once the eigenfunctions of the covariance $C(x, x_2, t_2)$ is obtained, we then need to determine the covariance function of the intermediate process $\sqrt{\lambda_i(x)}\xi_n(i, \theta)$ that corresponds to the eigenfunctions $f_i(t, x)$ in Eq. (44). According to Eq. (30), the covariance $R_i(x_1, t_1)$ can be easily derived as

$$R_i(x_1, t_1) = 1 - |x_1 - t_1|. \tag{46}$$

Thus, based on the analytical solution of the integral eigenvalue problem of type given in Eq. (43), the eigenvalues and eigenfunctions of $R_i(x_1, t_1)$ are given as

$$\tilde{\mu}_{ij} = \frac{2}{\omega_{ij}^2}, \quad \tilde{g}_{ij}(x) = \frac{\cos \omega_{ij} x}{\sqrt{\frac{1}{2} + \frac{\sin 2\omega_{ij}}{2\omega_{ij}}}} \tag{47}$$

where

$$\omega_{ij} = \omega_j = \begin{cases} j\pi, & j = 2k - 1 \\ \dfrac{2}{\tan \omega_j / 2}, & j = 2k. \end{cases} \tag{48}$$

The random variables $\tilde{\eta}_{ij}(\theta)$ that correspond to the eigenvalues $\tilde{\mu}_{ij}$ can be determined by using the method given in Eq. (4). Thus, by substituting Eqs. (44) and (47) into Eq. (39), the random field $w(x, t, \theta)$ is represented by

$$\hat{w}(x, t, \theta) = 2 \sum_{i=1}^{M_1} \sum_{k=1}^{M_2} \tilde{\eta}_{ij}(\theta) \frac{\sqrt{\omega_i} \cos \omega_i t \cos \omega_j x}{\sqrt{\omega_j \left(\omega_i + \sin 2\omega_i\right) \left(\omega_j + \sin 2\omega_j\right)}}. \tag{49}$$

In this example, the number of the terms that retained in Eq. (49) is adopted as $M_1 = 5$, and $M_2 = 5$. Fig. 2a and b show the first 6 eigenfunctions $f_i(t, x)$, $i = 1, \ldots, 6$ and eigenvalues associated with the covariance kernel as given in Eq. (41). Since the eigenfunctions $\tilde{g}_{ij}(x)$ and eigenvalues $\tilde{\mu}_{ij}$ of covariance $R_i(x_1, t_1)$ have the same structure as those given in Eq. (44), they also can be described by Fig. 2a and b. Figs. 3–5 show the exact covariance function, the approximated covariance, and the associated errors at the position $x = x_1 = t_1 = 0.5$, $t = t_1 = t_2 = 0.7$, and $x_1 = 0.2$, $x_2 = 0.6$, respectively. It can be seen that for all these cases, the approximations of covariance function based on Eq. (40) agree well with the exact covariance, illustrating the effectiveness of the proposed method.

We emphasize that, in this example the random field $w(x, t, \theta)$ can be also represented by the traditional K–L expansion since the covariance function in Eq. (41) is separable. If the traditional K–L expansion is used to simulate $w(x, t, \theta)$, the following two-dimensional Fredholm integral equation

$$\int_{\mathcal{D}} C(x_1, x_2, t_1, t_2) f_n(x_1, t_1) dx_1 dt_1 = \lambda_n f_n(x_2, t_2) \tag{50}$$

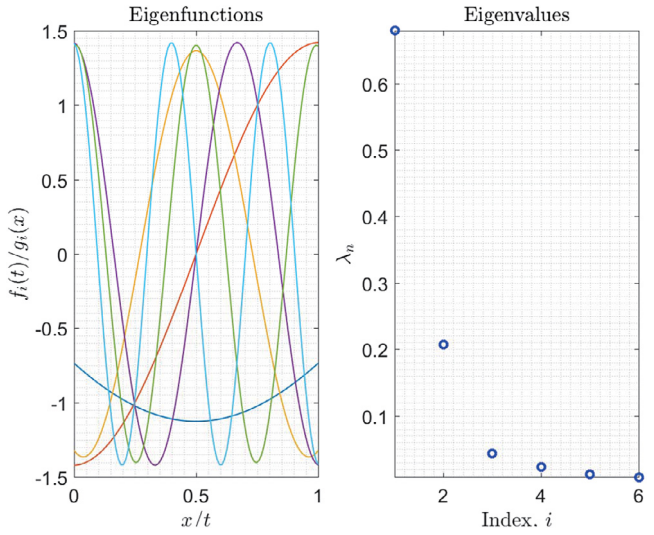


Fig. 2. The first six eigenfunctions and eigenvalues of the covariance in example 1.

has to be solved. According to [33], we assume that the eigenvalues and eigenfunctions are also separate, that is,

$$f_n(x_1, x_2) = f_i^{(1)}(x_1) f_j^{(2)}(x_2) \qquad \lambda_n = \lambda_i^{(1)} \lambda_j^{(2)}. \tag{51}$$

Then, Eq. (50) becomes

$$\lambda_i^{(1)} f_i^{(1)}(x_1) \lambda_j^{(2)} f_j^{(2)}(x_2) = \int_{\mathcal{D}_1} (1 - |x_1 - t_1|)\, f_i^{(1)}(t_1) dt_1 \int_{\mathcal{D}_2} (1 - |x_2 - t_2|)\, f_j^{(2)}(t_2) dt_2 \tag{52}$$

which is equivalent to

$$\begin{cases} \lambda_i^{(1)} f_i^{(1)}(x_1) = \displaystyle\int_{\mathcal{D}_1} (1 - |x_1 - t_1|)\, f_i^{(1)}(t_1) dt_1 \\ \lambda_j^{(2)} f_j^{(2)}(x_2) = \displaystyle\int_{\mathcal{D}_2} (1 - |x_2 - t_2|)\, f_j^{(2)}(t_2) dt_2. \end{cases} \tag{53}$$

By solving Eq. (53), the eigenfunctions are obtained as

$$f_i^{(1)}(x) = \frac{\cos \omega_i x}{\sqrt{\frac{1}{2} + \frac{\sin 2\omega_i}{2\omega_i}}}, \quad f_j^{(2)}(x) = \frac{\cos \omega_j x}{\sqrt{\frac{1}{2} + \frac{\sin 2\omega_j}{2\omega_j}}} \tag{54}$$

where

$$\omega_i = \begin{cases} i\pi, & i = 2k-1 \\ \dfrac{2}{\tan \omega_i/2}, & i = 2k, \end{cases} \qquad \omega_j = \begin{cases} j\pi, & j = 2k-1 \\ \dfrac{2}{\tan \omega_j/2}, & j = 2k. \end{cases} \tag{55}$$

Thus, the random field $w(x, t, \theta)$ can be also simulated by using the traditional K–L expansion as

$$\hat{w}(x, t, \theta) = 2 \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \eta_{ij}(\theta) \frac{\sqrt{\omega_i} \cos \omega_i t \cos \omega_j x}{\sqrt{\omega_j \left(\omega_i + \sin 2\omega_i\right)\left(\omega_j + \sin 2\omega_j\right)}}. \tag{56}$$

Note that Eq. (56) is identical to Eq. (49), again illustrating the validity of the proposed method.

4.2. Example 2: Non-stationary random field with separable covariance

It can be seen from Eq. (42) that, the quasi one-dimensional covariance does not contain variable x when we fix the position $x_1 = t_1 = x$. In order to examine a more general case, the next example considers a zero-mean

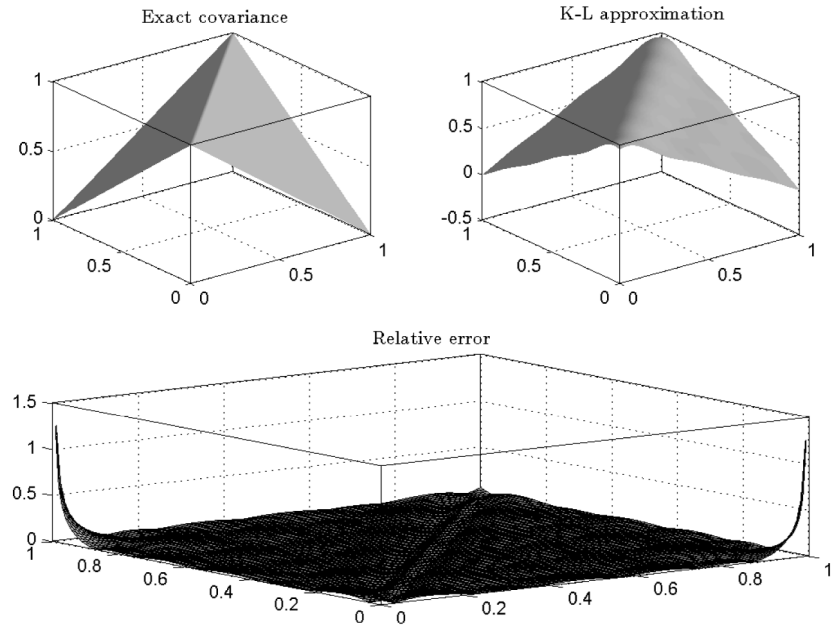


Fig. 3. Exact covariance, simulated covariance $C(x, x_2, t_2)$ and associated error at position $x = x_1 = t_1 = 0.5$.

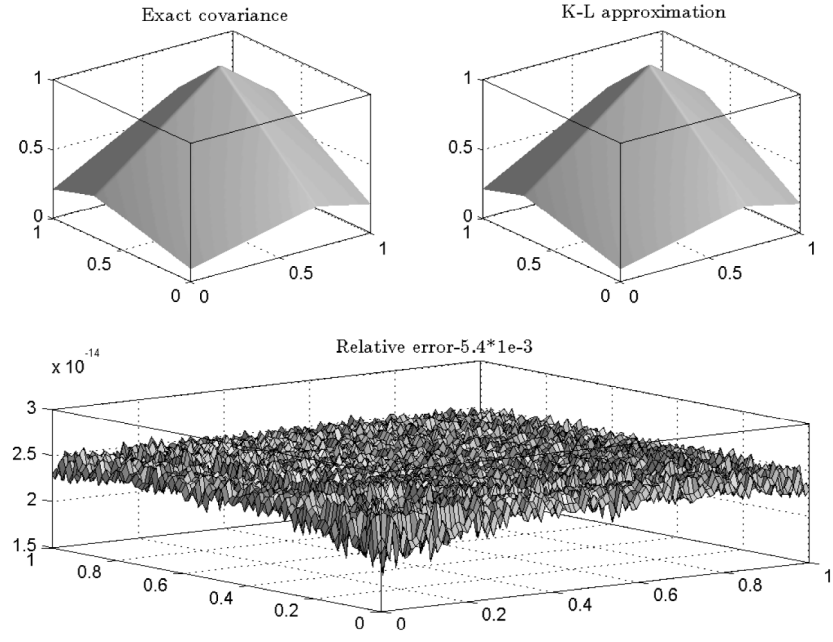


Fig. 4. Exact covariance, simulated covariance $C(t, x_1, x_2)$ and associated error at position $t = t_1 = t_2 = 0.7$.

two-dimensional random field $w(x, t, \theta)$ with covariance function given by

$$C(x_1, x_2, t_1, t_2) = \min(x_1, t_1) \min(x_2, t_2), x_i \in [0, 1], t_i \in [0, 1], \quad i = 1, 2. \tag{57}$$

Since the covariance is of the type of Wiener process, this example can be used to investigate the performance of the proposed method for simulating non-stationary random fields.

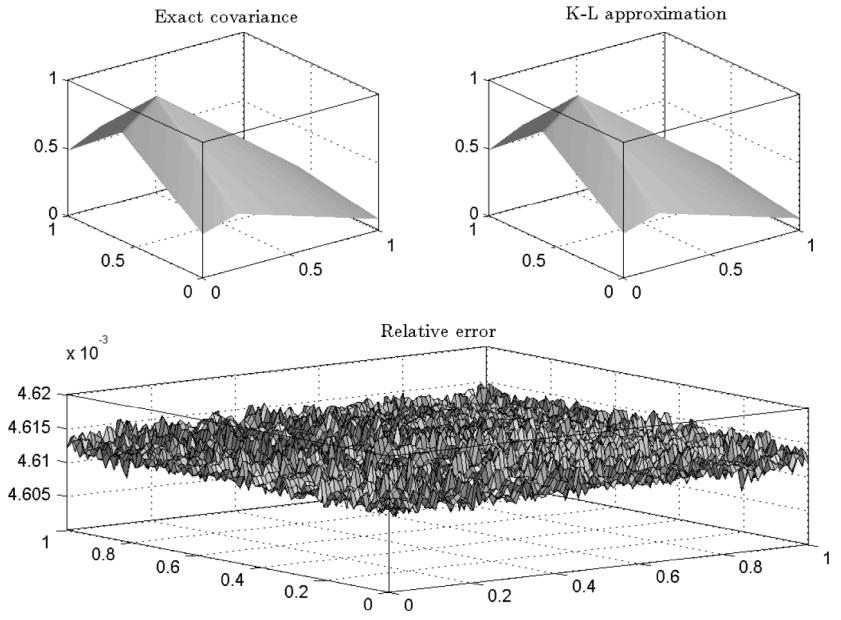


Fig. 5. Exact covariance, simulated covariance $C(x_1, x_2, t_1, t_2)$ and associated error at position $x_1 = 0.2$, $x_2 = 0.6$.

The first step of the proposed method is to fix $x_1 = t_1 = x$ to obtain a quasi one-dimensional covariance function as

$$C(x, x_2, t_2) = x \min(x_2, t_2), x_i \in [0, 1], t_i \in [0, 1], \quad i = 1, 2. \tag{58}$$

It can be seen that, different from example 1, variable x is involved in covariance $C(x, x_2, t_2)$. The analytical solution of the following Fredholm integral equation:

$$x \int_0^1 \min(x_2, t_2)\, f_n(x_2, x) dx_2 = \lambda_n(x) f_n(t_2, x) \tag{59}$$

is the eigenfunctions $f_n(t, x)$ and eigenvalues $\lambda_n(x)$ of covariance kernel $C(x, x_2, t_2)$. According to [10], $f_n(t, x)$ and $\lambda_n(x)$ can be analytically solved as

$$f_n(t, x) = \sqrt{2} \sin \omega_n t, \quad \lambda_n(x) = \frac{x}{\left[\left(n + \frac{1}{2}\right)\pi\right]^2} \tag{60}$$

where

$$\omega_n = \sqrt{\frac{x}{\lambda_n(x)}} = \left(n + \frac{1}{2}\right)\pi. \tag{61}$$

The next step is to determine the covariance $R_n(x_1, t_1)$ of the intermediate one-dimensional stochastic process. According to Eq. (30), the covariance $R_n(x_1, t_1)$ is derived as

$$\begin{aligned} R_n(x_1, t_1) &= \int \int_{\mathcal{D}_1(x_2)\times\mathcal{D}_1(t_2)} C(x_1, x_2, t_1, t_2) f_n(x_2) f_n(t_2) dx_2 dt_2 \\ &= 2 \min(x_1, t_1) \int_0^1 \int_0^1 \min(x_2, t_2) \sin \omega_n x_2 \sin \omega_n t_2 dx_2 dt_2 \\ &= \frac{1}{\omega_n^2} \min(x_1, t_1). \end{aligned} \tag{62}$$

Thus, based on Eq. (32), the eigenfunctions and eigenvalues of $R_n(x_1, t_1)$ are the solutions of the following Fredholm integral equation:

$$\frac{1}{\omega_n^2} \int_0^1 \min(x_1, t_1) \tilde{g}_{nk}(x_1) dx_1 = \tilde{\mu}_{nk} \tilde{g}_{nk}(t_1). \tag{63}$$

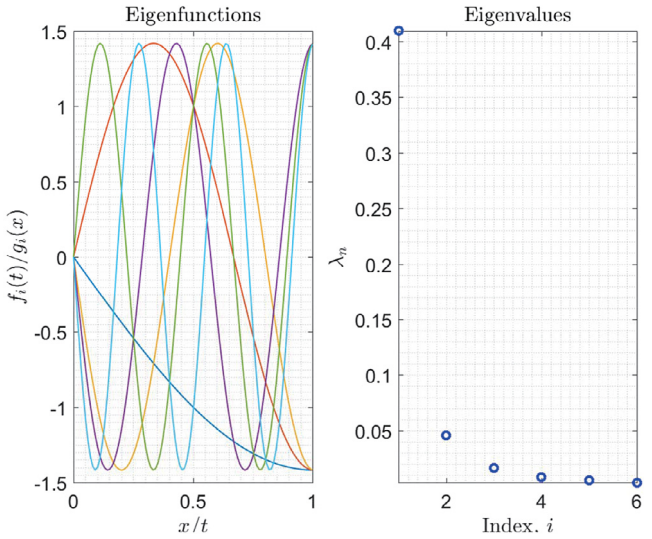


Fig. 6. The first six eigenfunctions and eigenvalues of the covariance in example 2.

Similar as the solution of Eq. (60), the closed-form solution of Eq. (63) are

$$\tilde{g}_{nk}(x) = \sqrt{2}\sin\omega_{nk}x, \quad \tilde{\mu}_{nk} = \frac{1}{\omega_n^2\left[\left(k+\frac{1}{2}\right)\pi\right]^2} = \frac{1}{\omega_n^2\omega_k^2} \tag{64}$$

where

$$\omega_{nk} = \omega_k = \left(k+\frac{1}{2}\right)\pi. \tag{65}$$

By substituting Eqs. (60) and (64) into Eq. (39), the random field $w(x,t,\theta)$ is consequently represented by

$$\hat{w}(x,t,\theta) = 2\sum_{i=1}^{M_1}\sum_{j=1}^{M_2}\tilde{\eta}_{ij}(\theta)\frac{1}{\omega_i^2\omega_j^2}\sin\omega_i t\sin\omega_j x \tag{66}$$

where $\tilde{\eta}_{ij}$ are the set of random variables that correspond to eigenfunctions $\tilde{g}_{nk}(x)$. Fig. 6a and b show the first 6 eigenfunctions $f_n(t,x), n = 1,\ldots,6$ and eigenvalues associated with the covariance kernel as given in Eq. (57). Since the eigenfunctions $\tilde{g}_{nk}(x)$ and eigenvalues $\tilde{\mu}_{nk}$ of covariance $R_n(x_1,t_1)$ have the same structure as those given in Eq. (60), they can be also described by using Fig. 6a and b. Figs. 7–9 show the exact covariance function, the approximated covariance, and the associated errors at the position $x = x_1 = t_1 = 0.5$, $t = t_1 = t_2 = 0.3$, and $x_1 = 0.3, x_2 = 0.8$, respectively. The number of the terms that retained in K–L expansion as in Eq. (66) is adopted as $M_1 = 5$, and $M_2 = 5$. Similar as the results in example 1, the approximations of covariance function based on Eq. (40) agree well with the exact covariance.

Note that this example demonstrates the necessity for using the covariance $R_n(x_1,t_1)$ in Eq. (30), instead of the covariance function $H_n(x_1,t_1)$ in Eq. (19) in the simulation of a multi-dimensional random field. According to Eq. (19), the covariance function $H_n(x_1,t_1)$ yields

$$H_n(x_1,t_1) = \frac{\min(x_1,t_1)}{\sqrt{x_1t_1}}. \tag{67}$$

If $H_n(x_1,t_1)$ is used as the covariance of the intermediate one-dimensional stochastic process, the eigenfunctions and eigenvalues of $H_n(x_1,t_1)$ cannot be readily obtained since the analytical solution of the corresponding Fredholm integral equation

$$\int_{\mathcal{D}_2}\frac{\min(x_1,t_1)}{\sqrt{x_1t_1}}g_{nk}(x_1)dx_1 = \mu_{nk}g_{nk}(t_1) \tag{68}$$

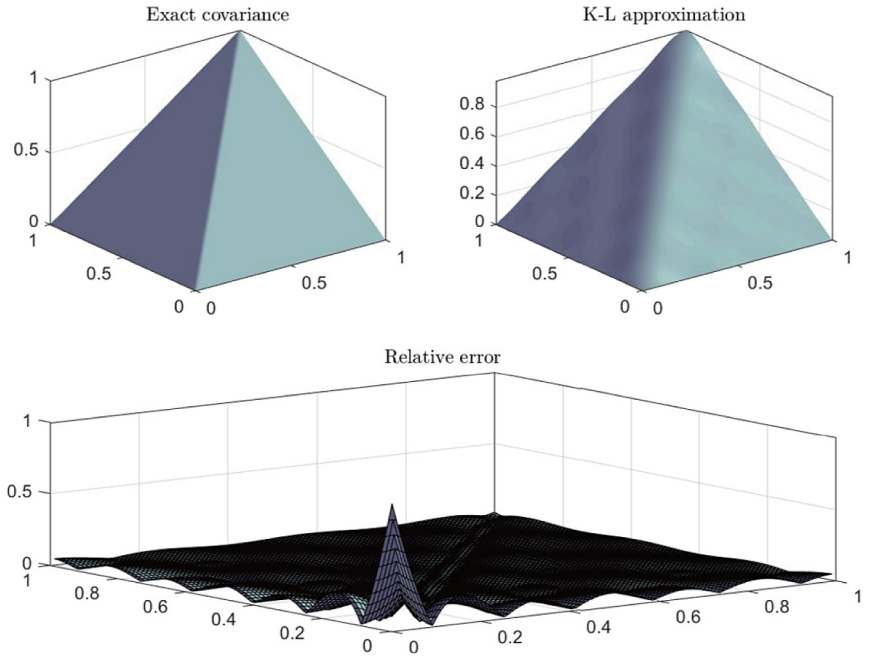


Fig. 7. Exact covariance, simulated covariance $C(x, x_2, t_2)$ and associated error at position $x = x_1 = t_1 = 0.5$.

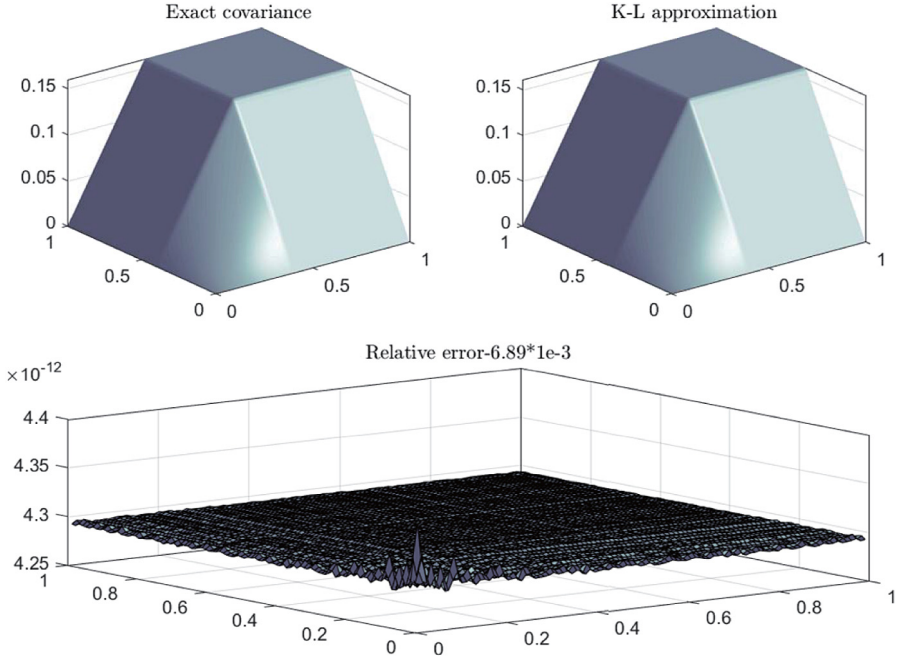


Fig. 8. Exact covariance, simulated covariance $C(t, x_1, x_2)$ and associated error at position $t = t_1 = t_2 = 0.3$.

is not available [10]. The main advantage for using the covariance of the intermediate process $\sqrt{\lambda_n(x)}\xi_n(x, \theta)$ is that the Fredholm integral equation of the covariance kernel $R_n(x_1, t_1)$ is, in general, easier to be solved.

We also emphasize that, although variable x is involved in the covariance $C(x, x_2, t_2)$, as shown in Eq. (58), the covariance function $C(x_1, x_2, t_1, t_2)$ that given in Eq. (57) is still separable. Therefore, one can simulate the random

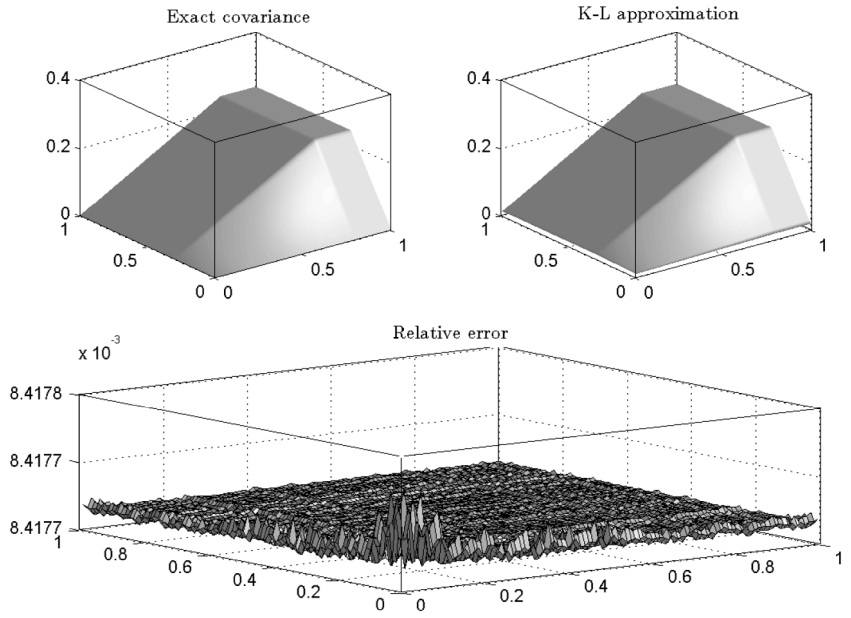


Fig. 9. Exact covariance, simulated covariance $C(x_1, x_2, t_1, t_2)$ and associated error at position $x_1 = 0.3$, $x_2 = 0.8$.

field using traditional K–L expansion by assuming that the eigenfunctions $f_n(x_1, x_2)$ and eigenvalues λ_n of covariance kernel $C(x_1, x_2, t_1, t_2)$ are also separable as,

$$f_n(x_1, x_2) = f_i^{(1)}(x_1) f_j^{(2)}(x_2), \quad \lambda_n = \lambda_i^{(1)} \lambda_j^{(2)}. \tag{69}$$

Thus, the following integral eigenvalue problem associated with the kernel $C(x_1, x_2, t_1, t_2)$

$$\int_{\mathcal{D}} C(x_1, x_2, t_1, t_2) f_n(x_1, t_1) dx_1 dt_1 = \lambda_n f_n(x_2, t_2) \tag{70}$$

is converted to

$$\lambda_i^{(1)} f_i^{(1)}(x_1) \lambda_j^{(2)} f_j^{(2)}(x_2) = \int_{\mathcal{D}_1} \min(x_1, t_1) f_i^{(1)}(t_1) dt_1 \int_{\mathcal{D}_2} \min(x_2, t_2) f_j^{(2)}(t_2) dt_2 \tag{71}$$

which is equivalent to

$$\begin{cases} \lambda_i^{(1)} f_i^{(1)}(x_1) = \displaystyle\int_{\mathcal{D}_1} \min(x_1, t_1) f_i^{(1)}(t_1) dt_1 \\ \lambda_j^{(2)} f_j^{(2)}(x_2) = \displaystyle\int_{\mathcal{D}_2} \min(x_2, t_2) f_j^{(2)}(t_2) dt_2. \end{cases} \tag{72}$$

The closed-form solution of Eq. (72) can be found in [10] as

$$f_i^{(1)}(x) = \sqrt{2} \sin \omega_i x, \quad \lambda_i^{(1)}(x) = \frac{1}{\left[\left(i + \frac{1}{2}\right)\pi\right]^2} \tag{73}$$

and

$$f_j^{(2)}(x) = \sqrt{2} \sin \omega_j x, \quad \lambda_j^{(2)}(x) = \frac{1}{\left[\left(j + \frac{1}{2}\right)\pi\right]^2} \tag{74}$$

where

$$\omega_i = \left(i + \frac{1}{2}\right)\pi, \quad \omega_j = \left(j + \frac{1}{2}\right)\pi. \tag{75}$$

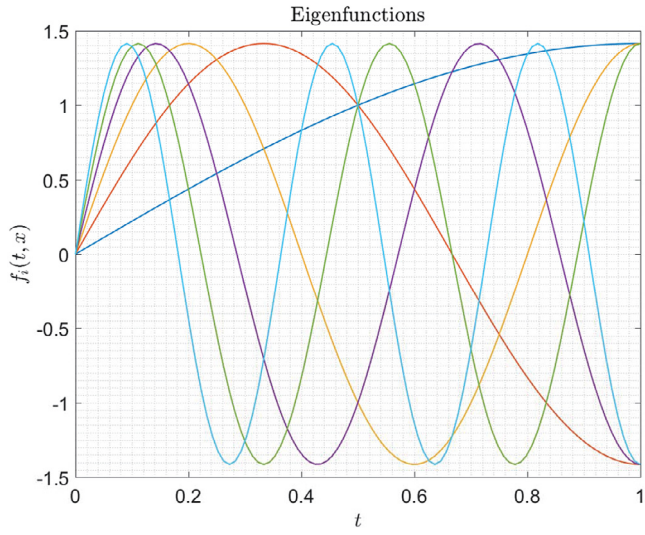


Fig. 10. The first six eigenfunctions $f_n(t, x)$ of the covariance in example 3.

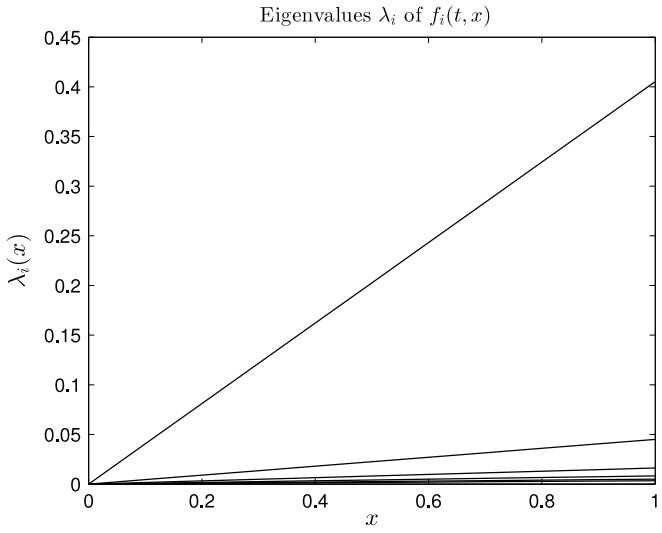


Fig. 11. The first six eigenvalue functions $\lambda_n(x)$ of the covariance in example 3.

Once the eigenfunctions $f_n(x_1, x_2)$ and eigenvalues λ_n of covariance kernel $C(x_1, x_2, t_1, t_2)$ are obtained, it will not be difficult to derive the same representation formula of the random field $w(x, t, \theta)$ as Eq. (66) that is given by the proposed method. Hence, the proposed method derives the same representation of the random field with separable covariance structure as the traditional K–L expansion, again demonstrating the application of the proposed method.

4.3. Example 3: Random field with non-separable covariance

It has been shown, in the first two examples, that proposed method achieves the same representation of a random field with a separable covariance, i.e., $C(x_1, x_2, t_1, t_2) = C(x_1, t_1)C(x_2, t_2)$, with traditional K–L expansion. In practice, if the covariance function is not separable, then the traditional K–L expansion does not work. In this example, we investigate the performance of the proposed method for handling such a problem. Consider a zero-mean

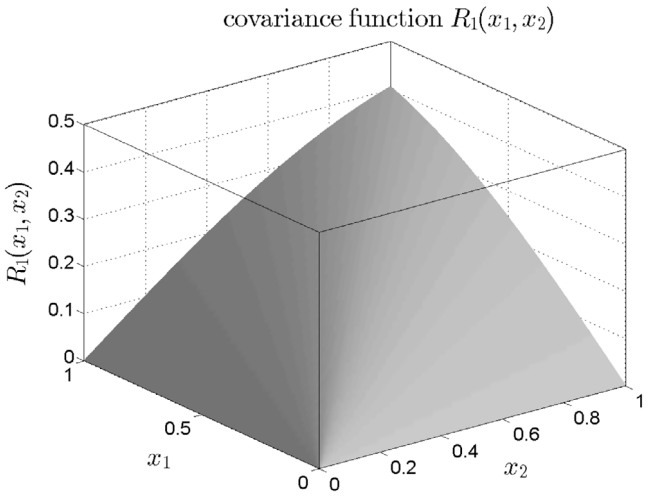


Fig. 12. Covariance function $R_1(x_1, x_2)$ of the first expanded intermediate process $\sqrt{\lambda_1(x)}\xi_1(x, \theta)$.

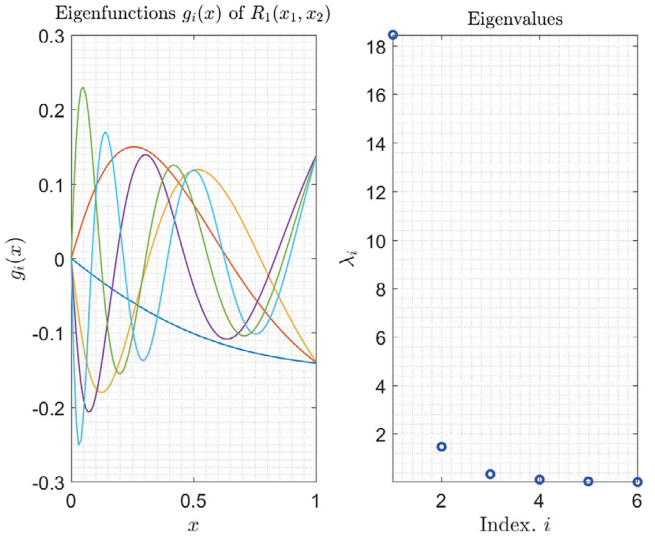


Fig. 13. The first six eigenfunctions and eigenvalues of the covariance $R_1(x_1, x_2)$.

two-dimensional random field $w(x, t, \theta)$ with covariance function given by

$$C(x_1, x_2, t_1, t_2) = \min(x_1 t_1, x_2 t_2), x_i \in [0, 1], t_i \in [0, 1], \quad i = 1, 2. \tag{76}$$

To simulate $w(x, t, \theta)$, we firstly fix $x_1 = x_2 = x$ to obtain a quasi one-dimensional covariance function $C(x, t_1, t_2)$ as

$$C(x, t_1, t_2) = x \min(t_1, t_2), x_i \in [0, 1], t_i \in [0, 1], \quad i = 1, 2. \tag{77}$$

Note that the covariance $C(x, t_1, t_2)$ involves variable x. The eigenfunctions and eigenvalues of covariance kernel $C(x, t_1, t_2)$ can be obtained by solving the following Fredholm integral equation:

$$x \int_0^1 \min(t_1, t_2) f_n(t_1, x) dt_1 = \lambda_n(x) f_n(t_2, x) \tag{78}$$

[10] provided the closed-form solution of Eq. (78) as

$$f_n(t, x) = \sqrt{2} \sin \omega_n t, \quad \lambda_n(x) = \frac{x}{\left[\left(n + \frac{1}{2}\right)\pi\right]^2} \tag{79}$$

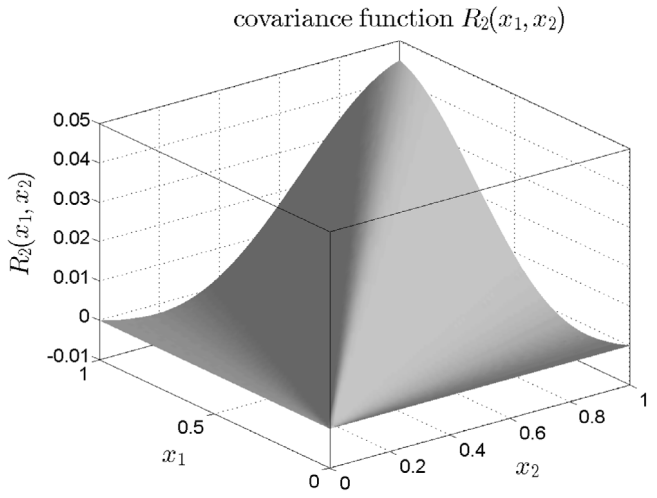


Fig. 14. Covariance function $R_2(x_1, x_2)$ of the second expanded intermediate process $\sqrt{\lambda_2(x)}\xi_2(x, \theta)$.

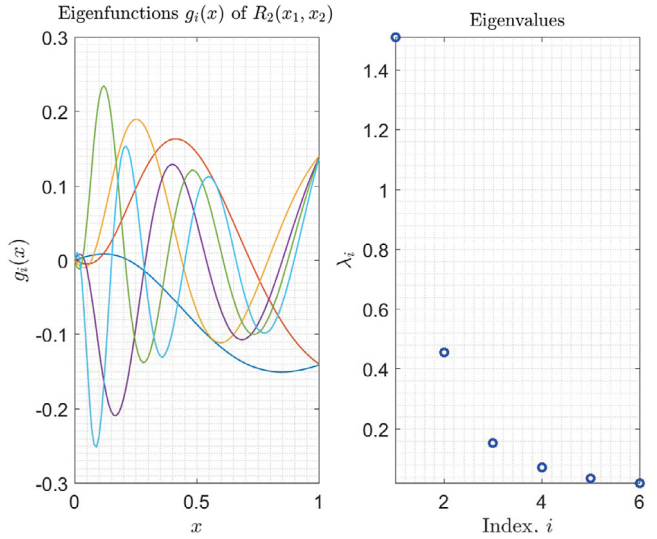


Fig. 15. The first six eigenfunctions and eigenvalues of the covariance $R_2(x_1, x_2)$.

where

$$\omega_n = \sqrt{\frac{x}{\lambda_n(x)}} = \left(n + \frac{1}{2}\right)\pi. \tag{80}$$

The next step is to compute the covariance of the intermediate one-dimensional process $\sqrt{\lambda_n(x)}\xi_n(x, \theta)$ as

$$\begin{aligned} R_n(x_1, x_2) &= \int\int_{\mathcal{D}_1 \times \mathcal{D}_1} C(x_1, x_2, t_1, t_2) f_n(t_1, x_1) f_n(t_2, x_2) dt_1 dt_2 \\ &= 2\int_0^1 \int_0^1 \min(x_1 t_1, x_2 t_2) \sin \omega_n t_1 \sin \omega_n t_2 dt_1 dt_2 \\ &= \frac{2(-1)^{n+1} x_1 x_2 \min(x_1, x_2) \cos\left[\omega_n (x_1/x_2)^p\right]}{\omega_n^3 p \left(x_1^2 - x_2^2\right)} \end{aligned} \tag{81}$$

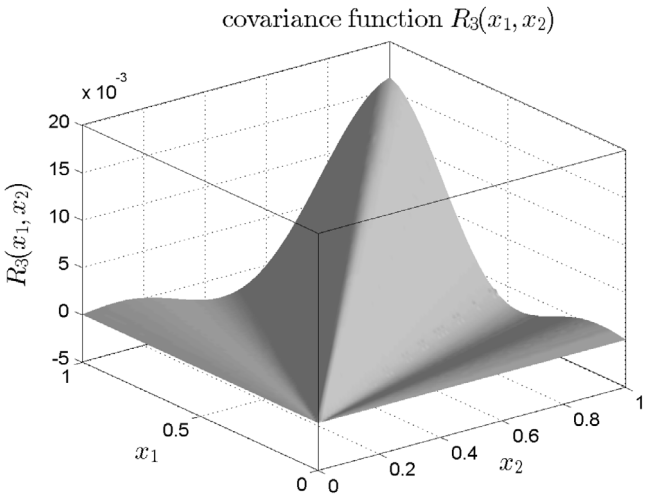


Fig. 16. Covariance function $R_3(x_1, x_2)$ of the third expanded intermediate process $\sqrt{\lambda_3(x)}\xi_3(x, \theta)$.

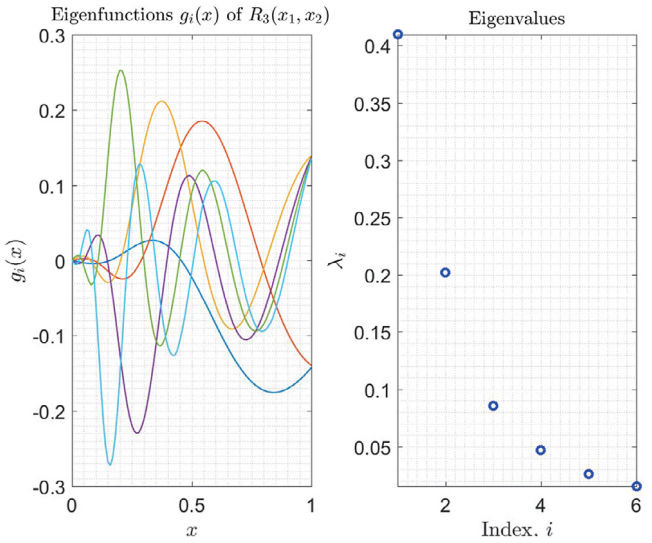


Fig. 17. The first six eigenfunctions and eigenvalues of the covariance $R_3(x_1, x_2)$.

where $p = \text{sign}(x_1 - x_2)$, and then to determine the eigenfunctions and eigenvalues of covariance $R_n(x_1, x_2)$ by solving the following integral eigenvalue problem

$$\int_{\mathcal{D}_2} R_n(x_1, x_2)\tilde{g}_{nk}(x_2)dx_2 = \tilde{\mu}_{nk}\tilde{g}_{nk}(x_1). \tag{82}$$

Due to the complexity of the form of $R_n(x_1, x_2)$ in Eq. (81), the analytical solution of Eq. (82) cannot be readily obtained. Nevertheless, the eigenfunctions $\tilde{g}_{nk}(x)$ and eigenvalues $\tilde{\mu}_{nk}$ can be obtained by the most used numerical methods, e.g., the Galerkin or wavelet-Galerkin method [23,24]. Once $\tilde{g}_{nk}(x)$ and $\tilde{\mu}_{nk}$ are determined, they can be used to simulate the random field by using Eq. (39). It should be noted that, the analytical form of the eigenfunctions $\tilde{g}_{nk}(x)$ is not necessary for the final representation of the random field as given in Eq. (39). The numerical solution of $\tilde{g}_{nk}(x)$ is also applicable for numerically representing the random field $w(x, t, \theta)$.

Figs. 10 and 11 show the first 6 eigenfunctions $f_n(t, x), n = 1, \ldots, 6$ and eigenvalues associated with the covariance kernel as given in Eq. (76). The number of the terms that retained in K–L expansion is adopted as $M_1 = 4$,

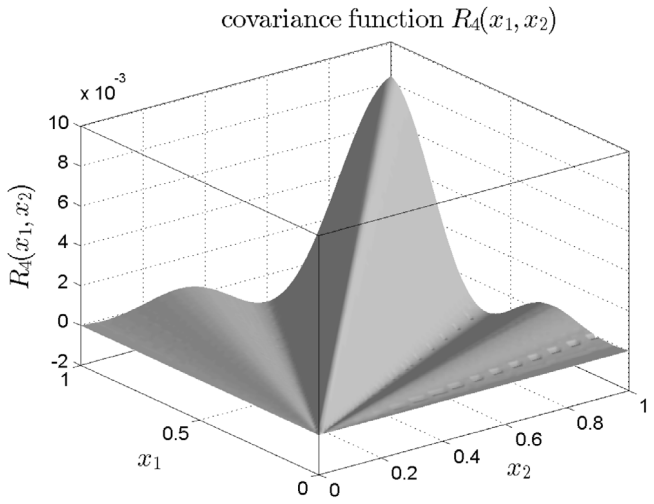


Fig. 18. Covariance function $R_4(x_1, x_2)$ of the third expanded intermediate process $\sqrt{\lambda_4(x)}\xi_4(x, \theta)$.

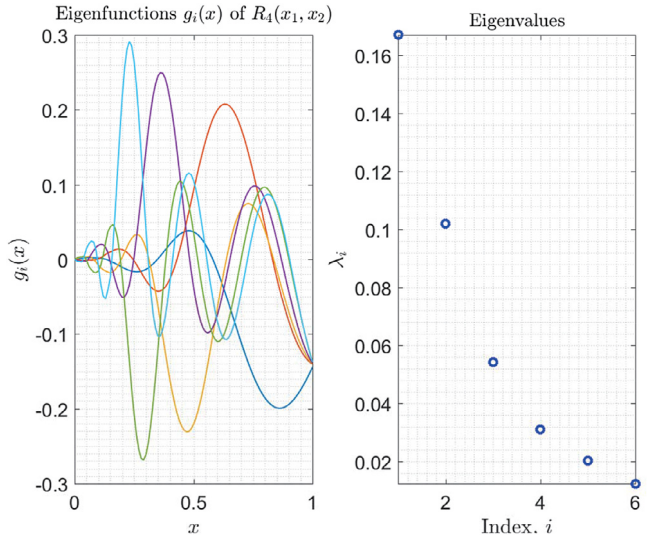


Fig. 19. The first six eigenfunctions and eigenvalues of the covariance $R_4(x_1, x_2)$.

and $M_2 = 5$ as in the previous examples. Figs. 12, 14, 16 and 18 show the covariance function $R_n(x_1, x_2)$, $n = 1, \ldots, 4$ of the first four expanded intermediate process $\sqrt{\lambda_n(x)}\xi_n(x, \theta)$, $n = 1, \ldots, 4$, respectively. It can be seen that the maximum value of $R_n(x_1, x_2)$ decreases rapidly with the increasing of n from 1 to 4. The maximum value of $R_4(x_1, x_2)$ has decreased to 0.008 and therefore it is no longer required to adopt $M_1 = 5$. Figs. 13, 15, 17 and 19 accordingly describe the first 6 eigenfunctions $g_i(t, x)$, $i = 1, \ldots, 6$ and eigenvalues associated with the covariance $R_n(x_1, x_2)$, $n = 1, \ldots, 4$, respectively. Fig. 20 shows the exact covariance function, the approximated covariance, and the associated errors at the position $x_1 = 0.4$, $x_2 = 0.7$. Similar observations can be found as the previous examples, the approximations of covariance function based on Eq. (40) generally agree well with the exact covariance.

In order to further demonstrate the proposed method, the traditional K–L expansion is also employed to simulate the random field in this example. In this study, the Galerkin scheme is used to numerically solve the two-dimensional Fredholm integral equation of the covariance in Eq. (76), where a set of fundamental algebraic polynomials with the power of n are adopted to represent the eigenfunctions of the covariance in Eq. (76). Obviously, the accuracy of the K–L representation depends on the number of the polynomials adopted, i.e., the number of n. Figs. 21 and 22

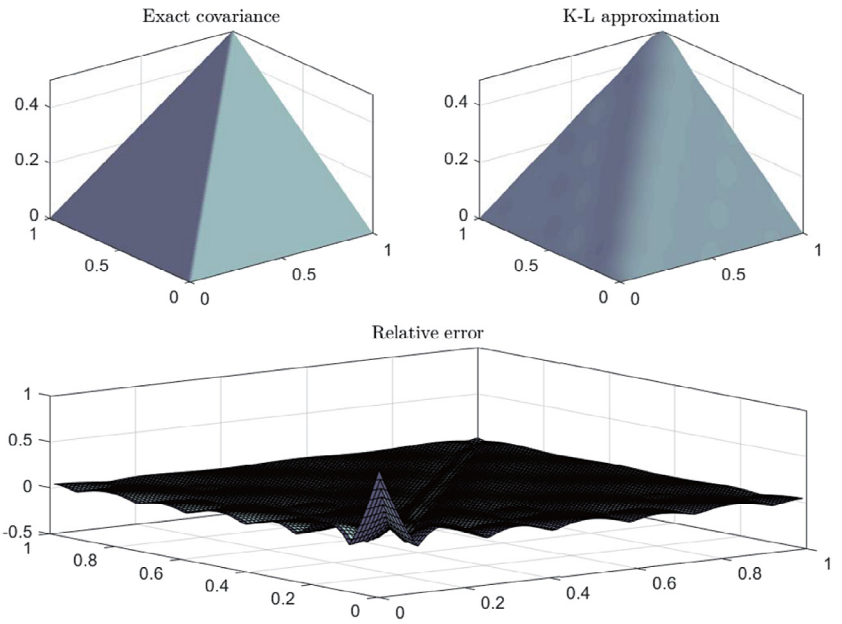


Fig. 20. Exact covariance, simulated covariance $C(x_1, x_2, t_1, t_2)$ and associated error at position $x_1 = 0.4$, $x_2 = 0.7$.

show the numerical results of the first 6 eigenvalues and eigenfunctions associated with the covariance in Eq. (76) when the highest power of the polynomial $p_n(\mathbf{x})$ is adopted as 40. The simulated covariance function at the position $x_1 = 0.4$, $x_2 = 0.7$ resulting from the K–L expansion with different choice of n, i.e., n adopts 26, 30, 34, 40, 50 and 60 is shown in Fig. 23. Figs. 24 and 25 compare the exact covariance, the approximated covariance from the proposed method and the traditional K–L expansion with different choice of n, and the associated errors at the same position. Obviously, the proposed method achieves good agreement with the exact covariance just by retaining the number of terms in Eq. (39) as $M_1 = 4$ and $M_2 = 5$. For the traditional K–L expansion method, the approximation error is obvious when the highest power of the polynomial n adopts 26, 30 and 34. Satisfied results can be achieved only when n is greater than 40. Note that although a large n will give a better approximation for the covariance in Eq. (76), it requires more computational cost in the simulation of the random field.

4.4. Example 4: Random field with exponential covariance

The last example investigates a special case in the context of non-separable covariance function of the random field. Consider a zero-mean two-dimensional random field $w(x, t, \theta)$ with covariance function given by

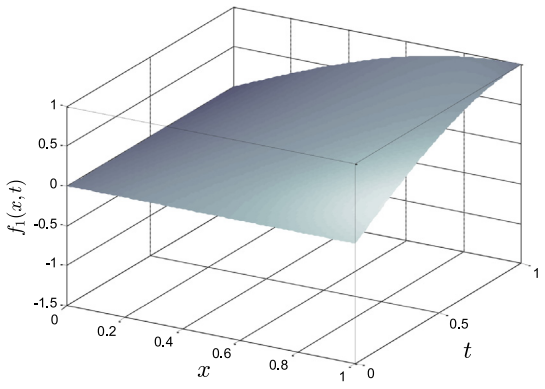
$$C(x_1, x_2, t_1, t_2) = e^{-|x_1 - t_1||x_2 - t_2|},\ x_i \in [-1, 1],\ t_i \in [-1, 1], \quad i = 1, 2. \tag{83}$$

If we fix $x_1 = t_1 = x$, then the 'quasi' one-dimensional covariance function $C(x, x_2, t_2)$ becomes one. Obviously, $C(x, x_2, t_2)$ cannot reflect the rate at which the correlation decays between variables x_2 and t_2. In this case, the proposed method does not work. In order to address this problem, we introduce a small positive number y such that $y = |x_1 - t_1| > 0$, and let

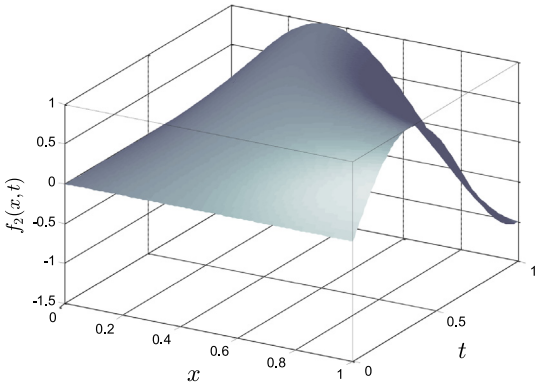
$$C(x, x_2, t_2) = e^{-y|x_2 - t_2|}. \tag{84}$$

Thus, the eigenfunctions and eigenvalues of $C(x, x_2, t_2)$ can be obtained by solving the following Fredholm integral equation

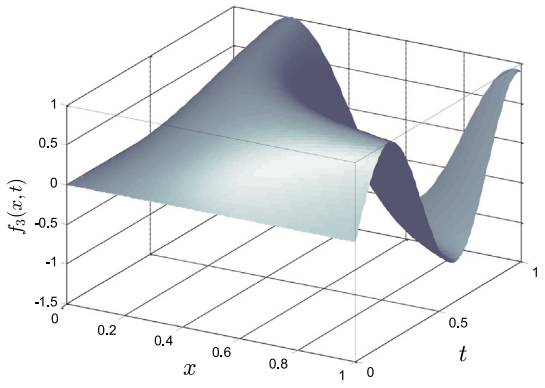
$$\int_{-1}^{1} e^{-y|x_2 - t_2|} f_n(x_2, y) dx_2 = \lambda_n(y) f_n(t_2, y). \tag{85}$$



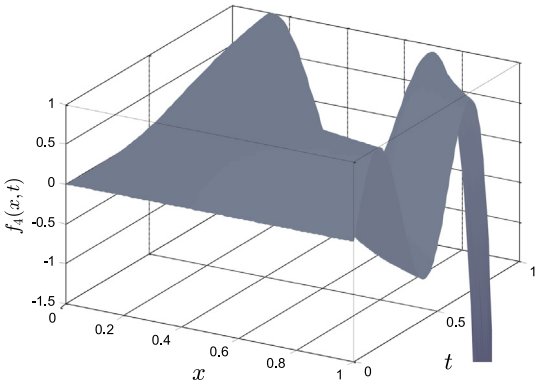
(a) Eigenfuntion $f_1(x,t)$ of covariance in Eq. (76).



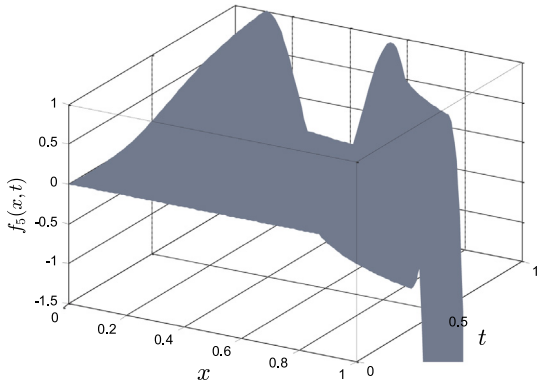
(b) Eigenfuntion $f_2(x,t)$ of covariance in Eq. (76).



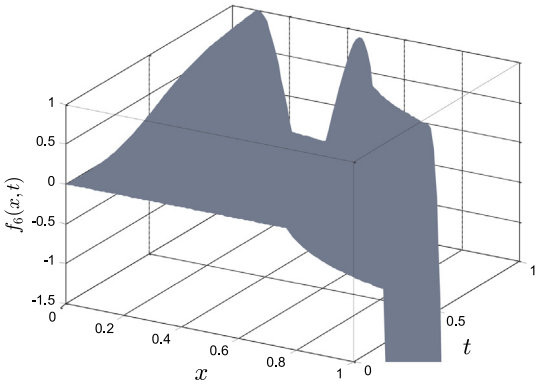
(c) Eigenfuntion $f_3(x,t)$ of covariance in Eq. (76).



(d) Eigenfuntion $f_4(x,t)$ of covariance in Eq. (76).



(e) Eigenfuntion $f_5(x,t)$ of covariance in Eq. (76).



(f) Eigenfuntion $f_6(x,t)$ of covariance in Eq. (76).

Fig. 21. Numerical results of the first six eigenfunctions $f_n(t,x)$ of the covariance in example 3.

The closed-form solution of Eq. (85) can be readily obtained from [10] as

$$f_n(x,y) = \frac{\cos\omega_n x}{\sqrt{1+(-1)^n \frac{\sin 2\omega_n}{2\omega_n}}} \tag{86}$$

where

$$\omega_n = \begin{cases} y/\tan\omega_n, & n = 2k \\ -y\tan\omega_n, & n = 2k-1. \end{cases} \tag{87}$$

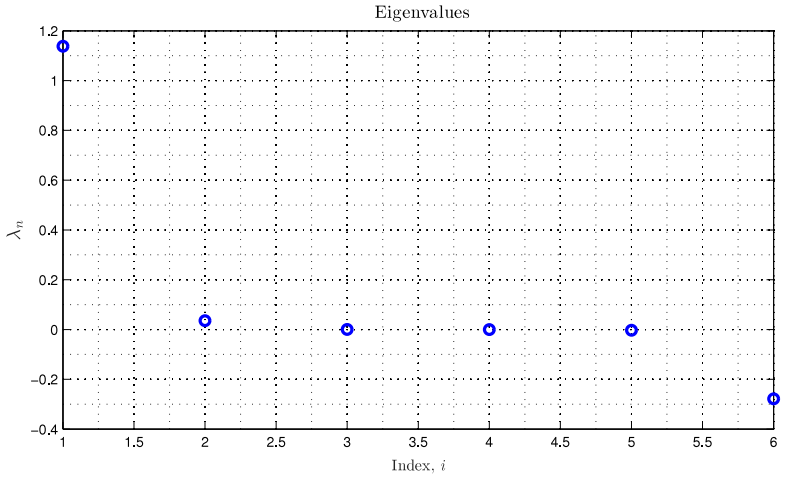


Fig. 22. Numerical results of the first six eigenvalues of the covariance in example 3.

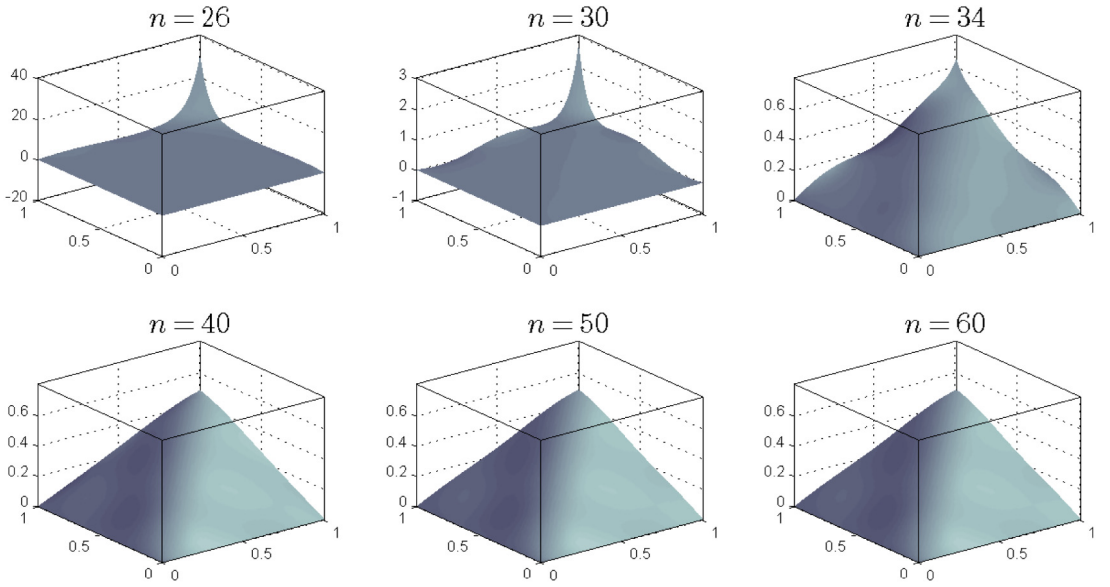


Fig. 23. K–L approximation of the covariance $C(x_1, x_2, t_1, t_2)$ at position $x_1 = 0.4$, $x_2 = 0.7$ with different choice of n.

The covariance of the intermediate one-dimensional process $\sqrt{\lambda_n(x)}\xi_n(x, \theta)$ is then computed as

$$R_n(y_1, y_2) = \int_{-1}^{1}\int_{-1}^{1} e^{-|y_1-y_2||x_2-t_2|} f_n(x_2, y_1)\, f_n(t_2, y_2)\, dx_2 dt_2. \tag{88}$$

Obviously, due to the complexity of the form of $R_n(y_1, y_2)$, the analytical solution of the integral eigenvalue problem associated with the covariance kernel $R_n(y_1, y_2)$ is difficult to solve, the Galerkin scheme is therefore adopted to approximate the eigenfunctions $\tilde{g}_{nk}(x)$ and eigenvalues $\tilde{\mu}_{nk}$ of the kernel $R_n(y_1, y_2)$. After $\tilde{g}_{nk}(x)$ and $\tilde{\mu}_{nk}$ are numerically determined, the random field $w(x, t, \theta)$ can be simulated by using Eq. (39). Fig. 26 shows the exact covariance, the approximated covariance and the associated approximation error. Obviously, the simulated covariance is in good accordance with the exact one even for this special case. It should be noted that when using Galerkin scheme to approximate the solution of the integral eigenvalue problem of the covariance kernel $R_n(y_1, y_2)$ at the set of discrete points (y_1^i, y_2^j), the number of the iterations of ω_n^i and ω_n^j in Eq. (87) is large. Nevertheless, the complexity of this problem arises from the adopted covariance kernel as given in Eq. (84), and it is irrelevant with the proposed method itself.

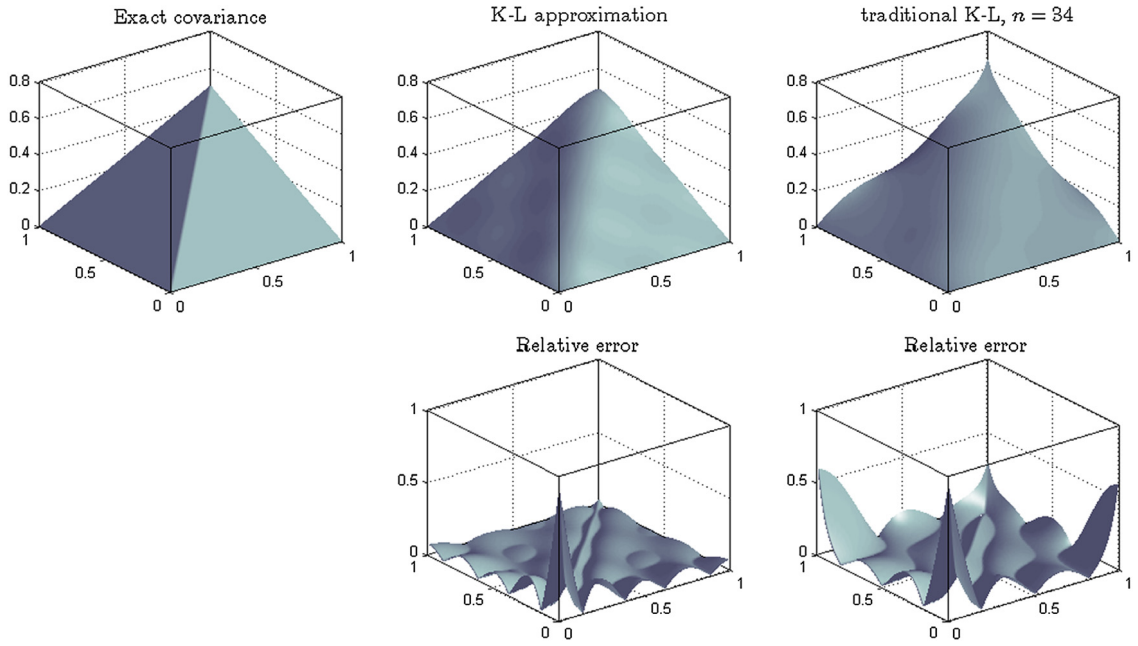


Fig. 24. Exact covariance, simulated covariance from the proposed and traditional K–L method with different choice of n.

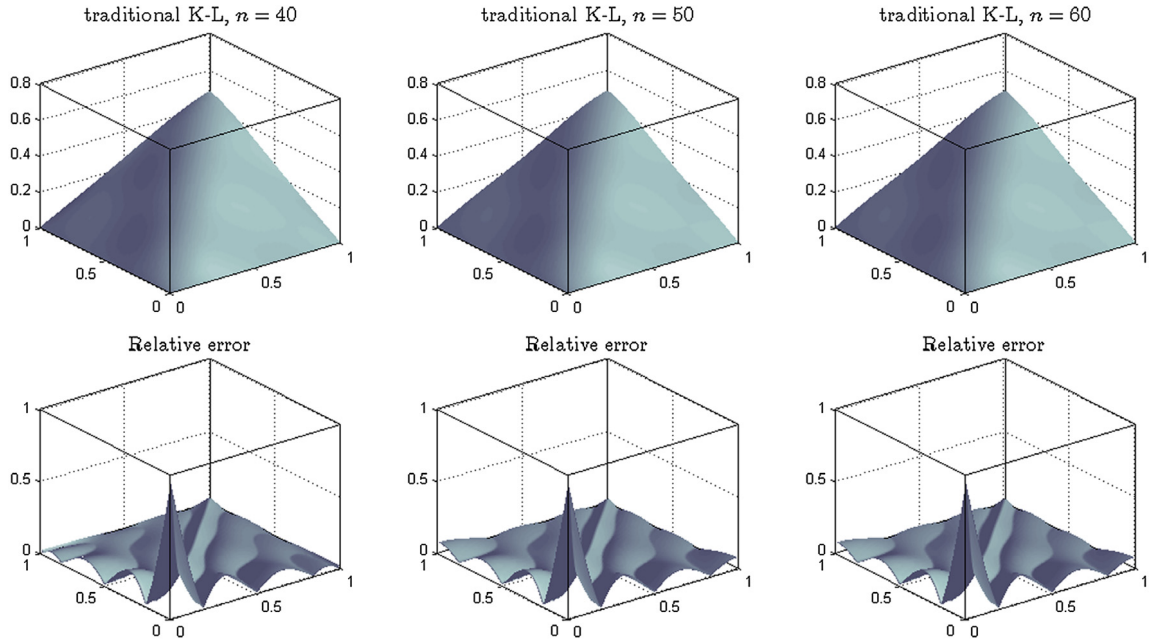


Fig. 25. Exact covariance, simulated covariance from the proposed and traditional K–L method with different choice of n.

5. Conclusion

This paper develops a new method for simulating multi-dimensional random fields by K–L expansion. The method expands an n-dimensional random field into a one-dimensional stochastic process and an $(n-1)$-dimensional random field, the obtained $(n-1)$-dimensional random field is further decomposed into a new one-dimensional stochastic

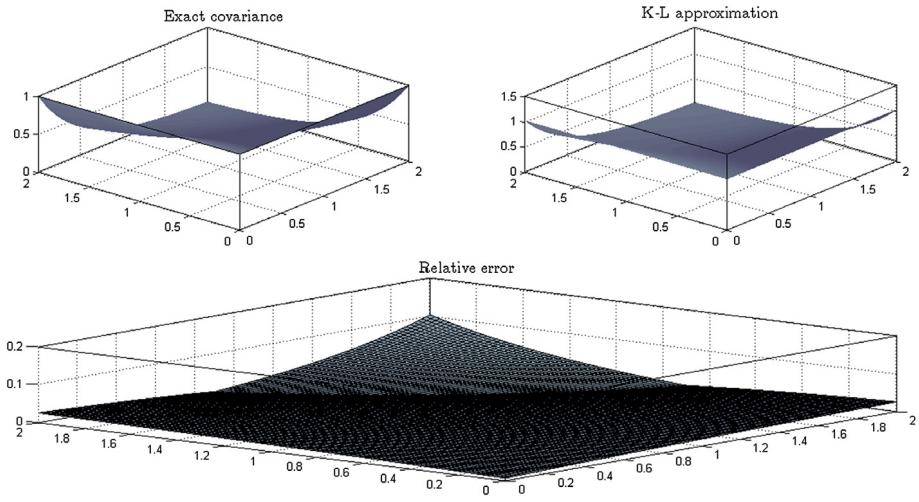


Fig. 26. Exact covariance, simulated covariance and the associated error in example 4.

process and an $(n-2)$-dimensional random field. By repeating this process, the original n-dimensional random field is decomposed into a total of n one-dimensional stochastic processes step by step, each can be represented by using the traditional K–L expansion. Thus, the developed method is embedded into the well-established framework of the K–L expansion for simulating stochastic process. The performance of the proposed method for simulating multi-dimensional random field is investigated by using four examples. It has been shown that for random field with a separable covariance structure, the proposed method provides the same representation of the random field with the traditional K–L expansion. For random field with non-separable covariance function, the approximation of the covariance given by the proposed method agrees well with the exact one, illustrating the effectiveness of the proposed method. We also emphasize that the proposed method possesses the potential for simulating multi-dimensional non-Gaussian random field, if it is used in conjunction with the method that proposed in [25].

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