Queuing Theory

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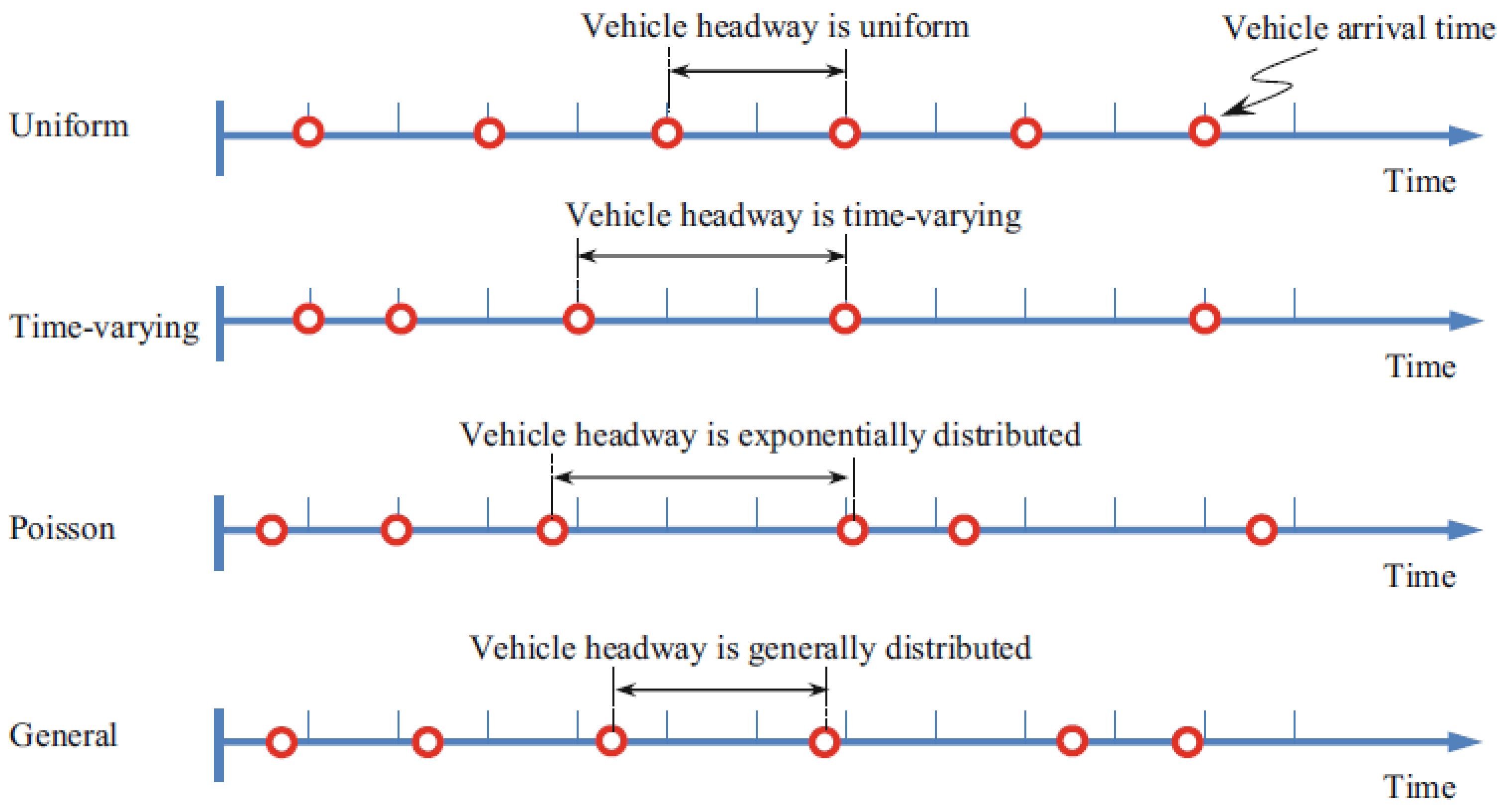
Arrival/Departure Processes at an Intersection

- Two broad categories of arrival processes:
  - Deterministic arrival:
    - ✓ The number and timing of vehicle arrival can be determined before the events take place.
    - ✓ If the arrival process is repeated multiple times, the result does not change from one experiment to another
  - Random arrival:
    - ✓ The number and timing of vehicle arrival cannot be determined before the events happen.
    - ✓ If the arrival process is repeated multiple times, the result may change from one experiment to another.

Poisson arrival process is of particular interest to the profession since it implies ***exponential*** headway distribution, a feature that gives rise to a set of statistically mature methods of determining delay.

Arrival/Departure Processes at an Intersection

- Focusses on four specific arrival processes:
  - Uniform,
  - Time-varying,
  - Poisson,
  - and general arrivals.



Uniform Arrival Process

- Assumes that arrival rate λ is constant
- Individual headways h_i are uniform and take the same value as the average headway h :

$$\lambda = \frac{q}{3600} \frac{\text{veh}}{\text{s}}$$

$$h_1 = h_2 = \cdots = h_i = \cdots = h = \frac{1}{\lambda} \frac{\text{s}}{\text{veh}}$$

Uniform Arrival Process

- ***Example***

  A stream of traffic with flow 900 veh/h arrives at an intersection approach starting from 9:00:00 am. If uniform arrival process is assumed, determine the arrival times for the first vehicles and calculate the time elapsed between two consecutive vehicles. How many vehicles would have arrived by 9:15:00 am?

Uniform Arrival Process

- ***Example***

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$$q = 900 \text{ veh/h} \rightarrow \lambda = 1/4 \text{ veh/s} \rightarrow h = 4 \text{ s/veh}$$

Vehicle ID, i	Arrival time, t_i	Headway, h_i (s)
1	9:00:00	Uniform
2	9:00:04	4
3	9:00:08	4
4	9:00:12	4
5	9:00:16	4
...	...	...

$$900 \frac{\text{veh}}{\text{h}} \times 15 \text{ min} = 900 \frac{\text{veh}}{\text{h}} \times \frac{1}{4} \text{ h} = 225 \text{ veh}$$

Time-Varying Arrival Process

- Assumes that arrival rate λ changes over time t.
- Individual headways h_i also change over time.
- ***Example***

  A stream of traffic arrives at an intersection approach starting from 9:00:00 am. If time-varying arrival process is assumed and the arrival rate in the next hour is given as:

$$\lambda(t) = \begin{cases} \frac{1}{8} + \frac{t}{7200} & \text{when } 0 \leq t < 1800\,\text{s} \\ \frac{5}{8} - \frac{t}{7200} & \text{when } 1800 \leq t < 3600\,\text{s} \end{cases}$$

  Determine the arrival times for the first five vehicles and calculate the time elapsed between two consecutive vehicles. How many vehicles will have arrived by 9:15:00 am?

Time-Varying Arrival Process

➢ ***Solution***

- We do not have a quick easy way to find arrival times and headways for a time-varying arrival process.
- We need to determine the cumulative number of vehicles arriving at this approach as a function of time.
- Then, look up the function and determine the timing when the 1st, 2nd, . . . vehicle arrives, respectively.

$$\lambda(t) = \begin{cases} \frac{1}{8} + \frac{t}{7200} & \text{when } 0 \le t < 1800\,\text{s} \\ \frac{5}{8} - \frac{t}{7200} & \text{when } 1800 \le t < 3600\,\text{s} \end{cases} \rightarrow A(\xi) = \begin{cases} \int_0^t \left(\frac{1}{8} + \frac{\xi}{7200}\right) d\xi & \text{when } 0 \le t < 1800\,\text{s} \\ \int_{1800}^t \left(\frac{5}{8} - \frac{\xi}{7200}\right) d\xi & \text{when } 1800 \le t < 3600\,\text{s} \end{cases} \rightarrow A(t) = \begin{cases} \frac{t}{8} + \frac{t^2}{14,400} & \text{when } 0 \le t < 1800\,\text{s} \\ \frac{5t}{8} - \frac{t^2}{14,400} - \boxed{900} & \text{when } 1800 \le t < 3600\,\text{s} \end{cases}$$

- 450 vehicles arrive in the first 30 minutes (1800s) – $A(t) = \ 225 \ + \ 225 \ = \ 450$
- 450 vehicles arrive in the second 30 minutes (1800 to 3600s) – $A(t) = \ 2250 \ - 900 \ - 900 = \ 450$
- 900 vehicles arriving during the hour

$$A(t) = 1 = \frac{t}{8} + \frac{t^2}{14400} \qquad t = 7.96\text{ s}$$

$t = 15.86\,s$ $\quad h = 7.9\ s$

$$A(t) = 2 = \frac{t}{8} + \frac{t^2}{14400} \quad t = 7.96\,s \qquad t = 15.86\ s$$

Vehicle ID, i	Arrival time, t_i	Headway, h_i (s)
1	9:00:00.00	Time-varying
2	9:00:07.96	7.96
3	9:00:15.86	7.90
4	9:00:23.69	7.76
5	9:00:31.45	6.42
…	…	…

Poisson Arrival Process

- Assumes that the probability of vehicles arriving during a time interval x follows Poisson distribution

$$P(n) = \frac{(\lambda x)^n e^{-\lambda x}}{n!}$$

- Where
  - $P(n)$ is the probability of having n vehicles arriving during time interval x,
  - λ is average rate of vehicles arriving at this approach per unit time, and

$$\lambda = \frac{q}{60}\,\frac{\text{veh}}{\text{min}} \quad \text{or} \quad \frac{q}{3600}\,\frac{\text{veh}}{\text{s}}$$

  - e is the base of natural logarithm, $e \approx 2.178.$

- A special property of Poisson distribution is that its mean is λt and variance is also λt

Poisson Arrival Process

- ***Example***
  A stream of traffic with flow 900 veh/h arrives at an intersection approach starting from 9:00:00 am. If Poisson arrival process is assumed and observation interval is 20 s, find the probability of having 0, 1, 2, . . ., 10, and more than 10 vehicles in an observation interval.

Poisson Arrival Process

- ***Example***

  A stream of traffic with flow 900 veh/h arrives at an intersection approach starting from 9:00:00 am. If Poisson arrival process is assumed and observation interval is 20 s, find the probability of having 0, 1, 2, . . ., 10, and more than 10 vehicles in an observation interval.

Arrival rate ➜ $\lambda = \dfrac{900\,\frac{\text{veh}}{\text{h}}}{3600\,\frac{\text{s}}{\text{h}}} = \dfrac{1}{4}\,\dfrac{\text{veh}}{\text{s}}$

➜ $\lambda x = \dfrac{1}{4} \times 20 = 5 \text{ veh}$ ➜

Number of vehicles observed, n	Probability of having n vehicles, $P(n)$
0	$P(0) = \frac{(5)^0 e^{-5}}{0!} \approx 0.007$
1	$P(1) = \frac{(5)^1 e^{-5}}{1!} \approx 0.034$
2	$P(2) = \frac{(5)^2 e^{-5}}{2!} \approx 0.084$
3	$P(3) = \frac{(5)^3 e^{-5}}{3!} \approx 0.140$
4	$P(4) = \frac{(5)^4 e^{-5}}{4!} \approx 0.175$
5	$P(5) = \frac{(5)^5 e^{-5}}{5!} \approx 0.175$
6	$P(6) = \frac{(5)^6 e^{-5}}{6!} \approx 0.146$
7	$P(7) = \frac{(5)^7 e^{-5}}{7!} \approx 0.104$
8	$P(8) = \frac{(5)^8 e^{-5}}{8!} \approx 0.065$
9	$P(9) = \frac{(5)^9 e^{-5}}{9!} \approx 0.036$
10	$P(10) = \frac{(5)^{10} e^{-5}}{10!} \approx 0.018$
>10	$P(10^+) = 1 - \sum_{n=0}^{5} \frac{(5)^n e^{-5}}{n!} \approx 0.014$

Poisson Arrival Process

- Poisson arrival process is of special interest to queuing analysis because it implies an interesting distribution of headways.
- A headway h > t means that no vehicles arrive during this time interval, i.e., $n = 0$.
- Therefore :

$$P(0) = P(h > x) \rightarrow P(0) = \frac{(\lambda x)^0 e^{-\lambda x}}{0!} = e^{-\lambda x}$$

$$P(h > x) = 1 - P(h \leq x) \rightarrow P(h \leq x) = 1 - e^{-\lambda x}$$

- Note that the above equation happens to be the cumulative distribution function of exponential distribution whose probability density function is:

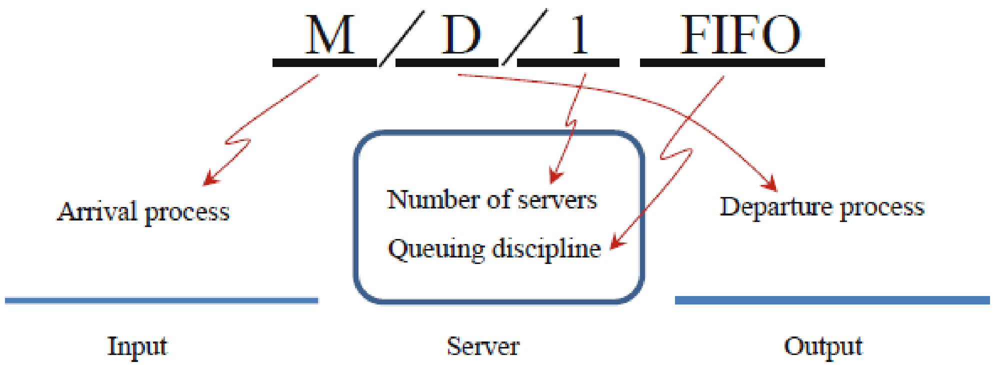
$$f(x) = \lambda e^{-\lambda x}$$

- A special property of exponential distribution is that its mean is $1/\lambda$ and variance is $1/\lambda^2$.

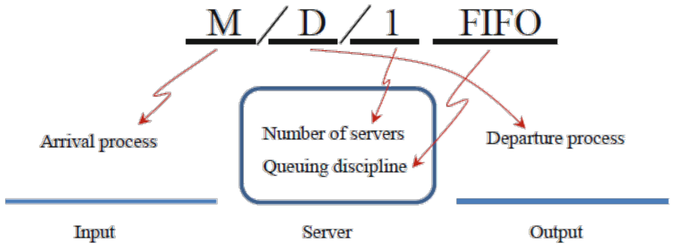
Poisson arrival process is equivalent to ***exponential distribution of headways*** with arrival rate $\lambda > 0$.

Basics of Queuing Theory

- A queuing system consists of three components: input, a server, and output.
  - The input represents customers (e.g., vehicles) arriving at the system.
  - Customers receive service at the server which takes some time.
  - At the end of service, customers depart from the system as the output.
- Therefore, a queuing system can be summarized by Kendall's notation as a four section code



Basics of Queuing Theory

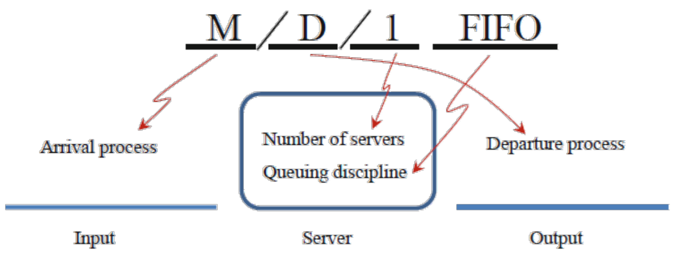


- The first section denotes customers' arrival process which is quantified by arrival rate γ or inter-arrival time h, and the second section specifies the departure process with the rate of μ.

Code	Name	Description
D	Deterministic	Deterministic arrival with actual arrival events predictable in advance. For example, uniform arrival or time-varying arrival.
M	Markovian	Random arrival that follows Poisson distribution or equivalently inter-arrival times (headways) follow exponential distribution
G	General	Random arrival that follows a distribution other than Poisson can be a known type of distribution or an empirical one.

- Uniform arrival has a constant λ or a constant h, where $h = 1/\lambda$.
- Time-varying arrival has $\lambda = \lambda(t)$ or $h = h(t)$.
- $Poisson$ arrival has number of vehicles arriving during time interval x, n, following $Poisson$ distribution or inter-arrival times, h, following $exponential$ distribution.
- General arrival has number of vehicles arriving during time interval x, n, or interarrival times, h, following general distribution with mean m and variance σ^2.

Basics of Queuing Theory



- This section specifies number of servers or channels available to customers.
- The last section deals with the queuing discipline of the system
  - First-in-first-out (FIFO) : customers are served in the order that they arrive at the system
  - Last in-first-out (LIFO) : customers are served in the reverse order to what they arrive at the system.
- For all traffic-oriented queues, the FIFO queuing discipline is the more appropriate.
- Typical questions to be answered in a queuing system are:
  - What is the time of queue dissipation?
  - What is the longest queue length?
  - What is the total delay experienced by all customers?
  - What is the average delay per customer?
  - What is the longest wait time of any customer?

Queuing System: D/D/1

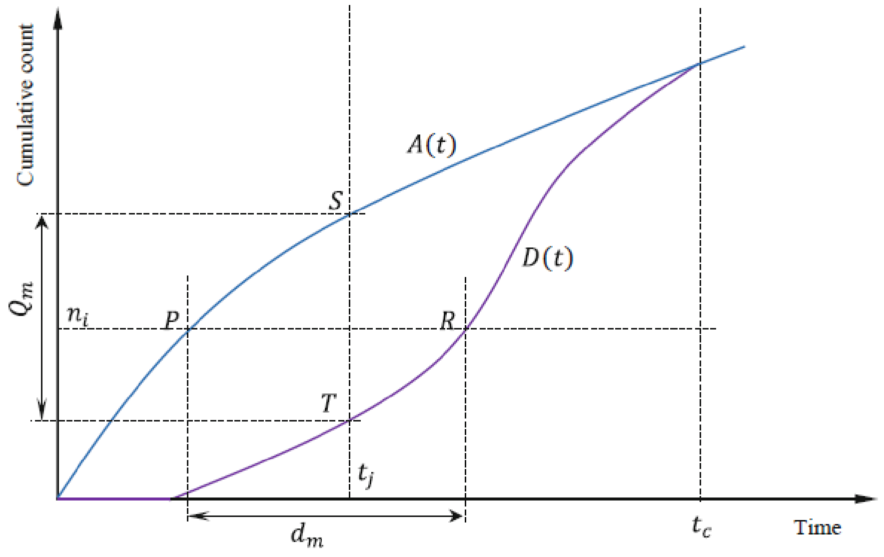
- If a queuing system has deterministic arrival and deterministic departure with one server operating under $FIFO$ principle, the best way to address the above questions is a graphical approach that constructs cumulative arrival curve $A(t)$ and cumulative departure curve $D(t)$ to help with the analysis.
- Assume that the deterministic arrival rate is $\lambda(t)$ and the deterministic departure rate is $\mu(t)$.
- The cumulative arrival curve $A(t)$ is:

$$A(t) = \int_0^t \lambda(\xi)d\xi$$

- Similarly, the cumulative departure curve $D(t)$ is:

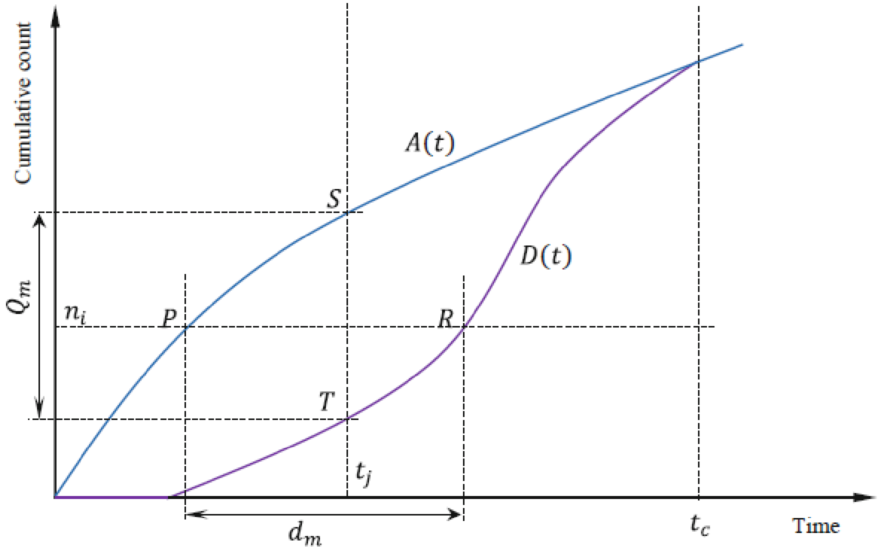
$$D(t) = \int_0^t \mu(\xi)d\xi$$

- A graphical representation of the system can be constructed



Queuing System: D/D/1

- The $A - curve$ represents cumulative number of customers arriving at the system as a function of time.
- The $D - curve$ with a delayed start represents cumulative number of customers departing from the system as a function of time.
- The horizontal distance between $A - curve$ and $D - curve$ (PR) represents the delay experienced by the corresponding vehicle.
- The vertical distance between $A - curve$ and $D - curve$ (ST) represents the queue length observed at the corresponding time (tj).
- The area bounded by $A - curve$, $D - curve$, and the two axes represents total delay experienced by all vehicles.
- The $D - curve$ should always be below the $A - curve$, or at most catches up with the $A - curve$.
- It means that the queue has just dissipated (at time t_c).



Queuing System Example: D/D/1

- ***Example***
  Vehicles arrive at the gate of a recreational park at a constant rate of 10 vehicles per minute starting from $8h00\ am$. The park opens at $8h30\ am$ and it takes each driver $4\ s$ to check-in and pass through the gate. If a $D/D/1$ queuing system is assumed, find queue clearance time, longest queue, total delay, average delay, and longest wait time.

Queuing System Example: D/D/1

➤ ***Solution***

➔ The arrival rate is given as a constant: $\lambda(t) = 10 \quad \text{for } t \geq 0$ $\quad t$ is time referenced from $8{:}00\ am$

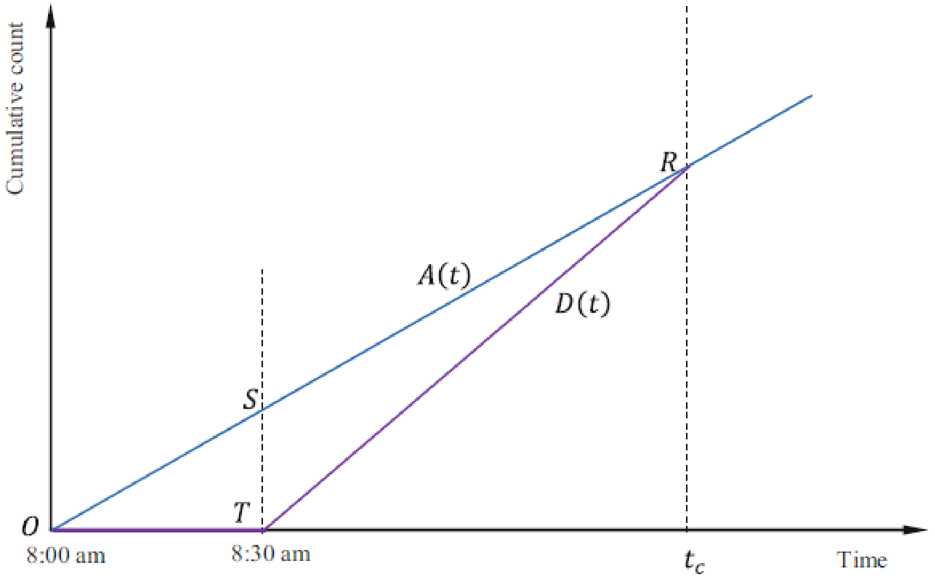
➔ The departure is given as a constant headway: $t(t) = 4 \text{ s/veh} = 15 \text{ veh/min}$

➔ The park opens 30 min after $8{:}00\ am$. Therefore, the departure rate is:
$$\mu(t) = \begin{cases} 0 & \text{for } 0 \leq t < 30 \\ 15 & \text{for } t \geq 30 \end{cases}$$

➔ The graph is constructed where the slopes of the $A - curve$ and $D - curve$ are 10 and 15 veh/min

$$A(t) = \int_0^t \lambda(\xi) d\xi = \int_0^t 10 d\xi = 10t$$

$$D(t) = \int_0^t \mu(\xi) d\xi = \begin{cases} \int_0^{30} 0 d\xi = 0 & \text{for } 0 \leq t < 30 \\ \int_0^{30} 0 d\xi + \int_{30}^t 15 d\xi = 15(t-30) & \text{for } t \geq 30 \end{cases}$$



Queuing System Example: D/D/1

- ***Solution (continued)***

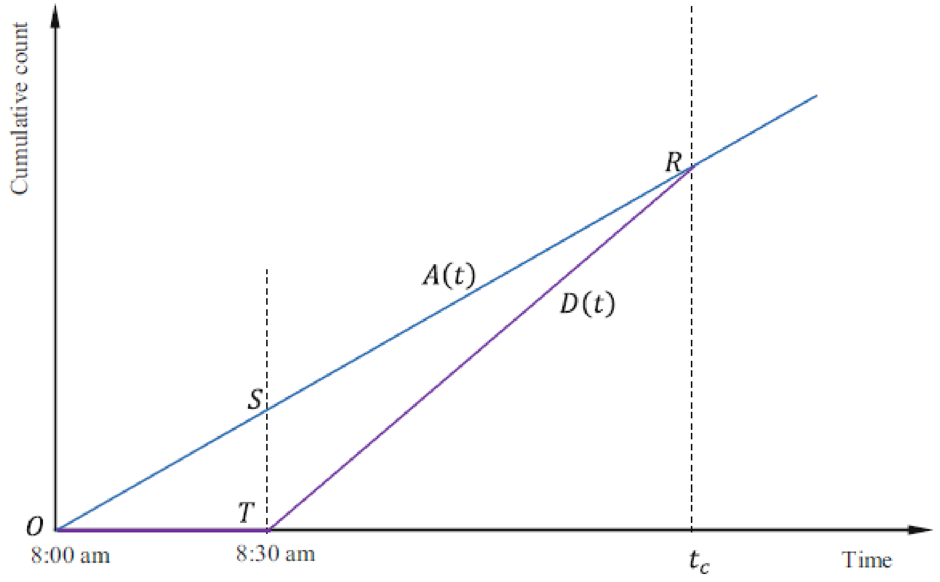
➜ **Queue clearance time:** $10t_c = 15(t_c - 30), \quad t_c = 90$ The queue clears 90 min after 8: 00 *am*, i.e., 9: 30 *am*.

➜ **Longest queue:** $\mathrm{ST} = 10 \times 30 = 300$ vehicles

the longest vertical distance between $A - curve$ and $D - curve$, which is found to be ST achieved at 8: 30 *am*.

➜ **Longest wait time:**

The longest wait time for any vehicle is the longest horizontal distance between $A - curve$ and $D - curve$, which is found to be OT achieved for the first vehicle:
$OT \ = \ 30 \text{ min}$



Queuing System Example: D/D/1

➢ ***Solution (continued)***

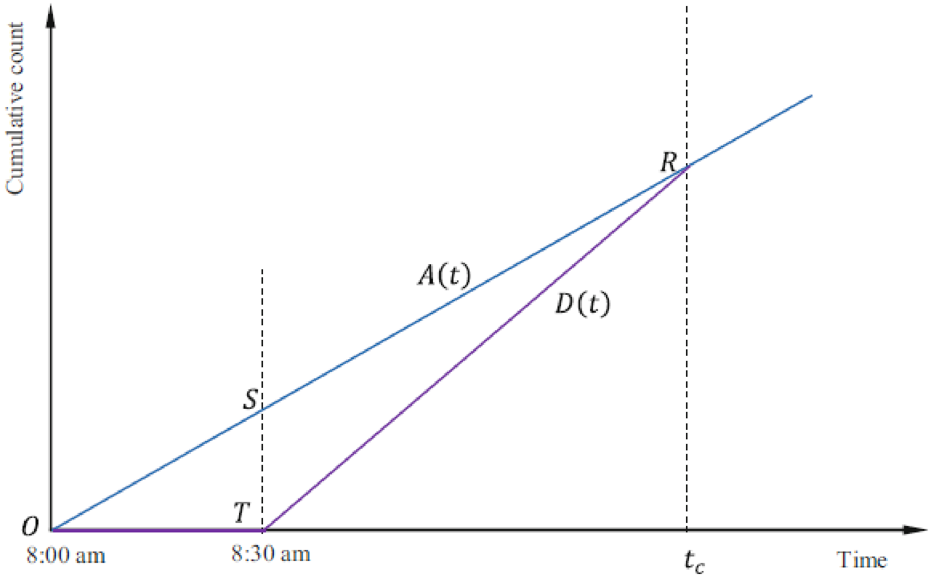
➜ **Total delay:** Area bounded by A-curve and D-curve

$$\Gamma = \int_0^t (A(\xi) - D(\xi))d\xi = \int_0^{30} (10\xi - 0)d\xi + \int_{30}^{90} [10\xi - 15(\xi - 30)]d\xi$$

$$= 4500 + 9000 = 13,500 \quad \text{veh min}$$

The total delay can also be determined as the sum of areas of triangles OTR:

$$\Gamma = S_{\text{OTR}} = \frac{1}{2} \times 30 \times (10 \times 90) = 13,500 \quad \text{veh min}$$



➜ **Average delay:**

There will be $N = 10 \times 90 = 900$ vehicles arrived at t_c. Therefore, the average delay per vehicle, W, is:

$$d = \frac{\Gamma}{N} = \frac{13,500}{900} = 15 \text{ min}$$

Queuing System Example: D/D/1 (time-varying)

- ***Example***
  Vehicles arrive at the gate of a recreational park at a rate of 10 vehicles per minute starting from 8:00 am. The park opens at 8:30 am and begins to admit customers. However, the gate attendant needs some time to warm up, after which he is able to speed up in handling admission according to a rate of 0.2x, where x is time elapsed from the start of admission. Again, a D/D/1 queuing system is assumed, find queue clearance time, longest queue, total delay, average delay, and longest wait time.

Queuing System Example: D/D/1 (time-varying)

- ***Solution***

→ The arrival rate is given as a constant: $\lambda(t) = 10 \quad \text{for } t \geq 0$ $\quad t$ is time referenced from $8{:}00\ am$

→ The departure is given as time-varying:

The park opens 30 min after $8{:}00\ am$.

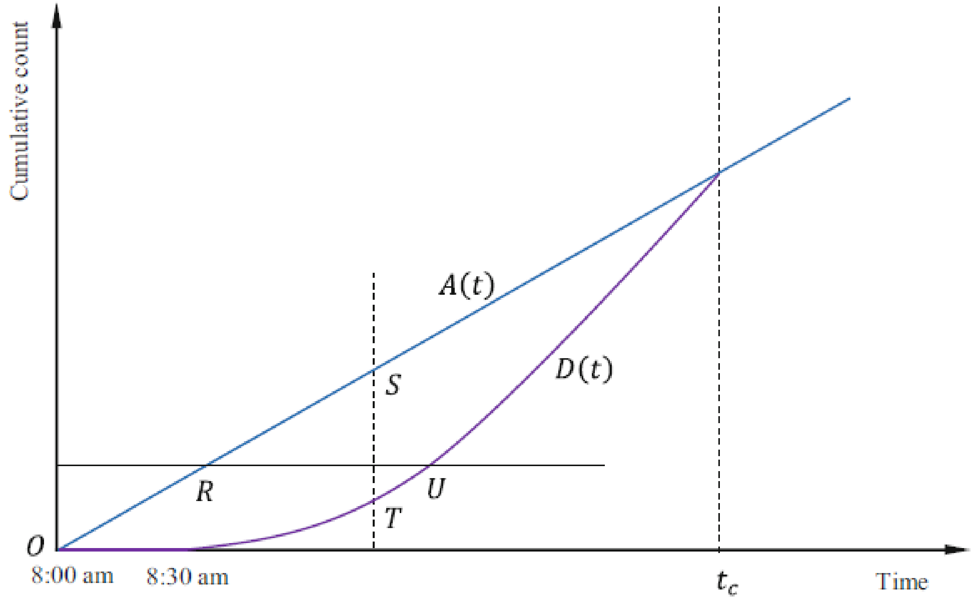
$$\mu(t) = \begin{cases} 0 & \text{for } 0 \leq t < 30 \\ 0.2(t-30) & \text{for } t \geq 30 \end{cases}$$

→ Cumulative arrival curve, the A-curve, and cumulative departure curve, the D-curve, are determined as follows

$$A(t) = \int_0^t \lambda(\xi)d\xi = \int_0^t 10d\xi = 10t$$

$$D(t) = \int_0^t \mu(\xi)d\xi$$

$$= \begin{cases} \int_0^{30} 0d\xi = 0 & \text{for } 0 \leq t < 30 \\ \int_0^{30} 0d\xi + \int_{30}^t 0.2(\xi - 30)d\xi = 0.1t^2 - 6t + 90 & \text{for } t \geq 30 \end{cases}$$



Queuing System Example: D/D/1 (time-varying)

- ***Solution (continued)***

➜ **<u>Queue clearance time:</u>** $10t_c = 0.1{t_c}^2 - 6t_c + 90, \quad t_c = 154.16$

The queue clears 154.16 min after $8{:}00\ am$ (about $10{:}34\ am$)

➜ **<u>Longest queue:</u>**

The longest vertical distance between $A-$curve and $D-$curve.

$$L(t) = \begin{cases} 10t & \text{for} 0 \le t < 30 \\ 10t - (0.1t^2 - 6t + 90) & \text{for} t \ge 30 \end{cases}$$

First Branch: the maximum value is achieved at $t = 30$ min

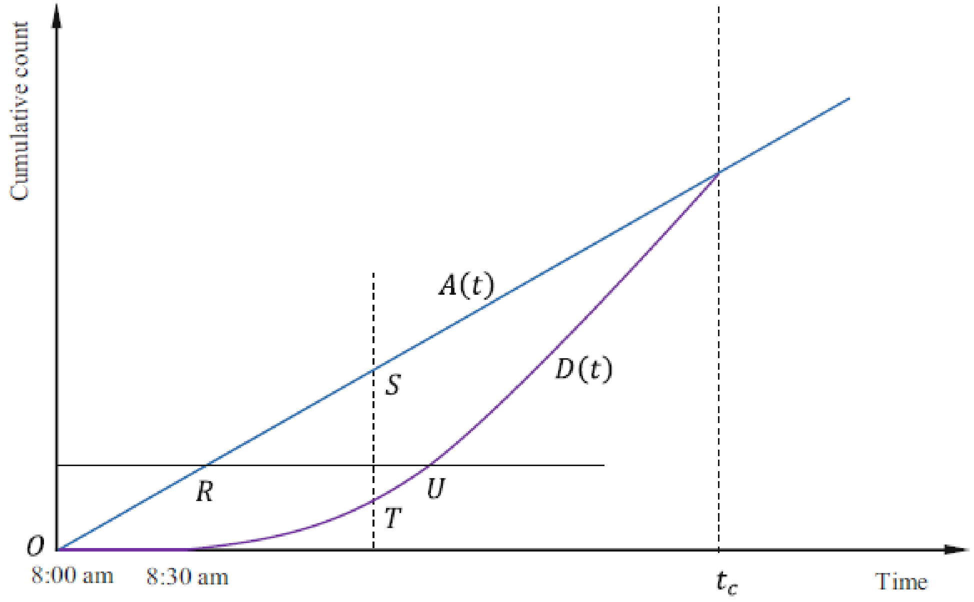
$$L(t)|_{t=30} = 300$$

Second Branch: the maximum value is achieved at $t = 80$ min

$$\frac{dL}{dt} = \frac{d(-0.1t^2 + 16t - 90)}{dt} = -0.2t + 16 = 0, \quad t = 80$$

$$L(t)|_{t=80} = 550$$

Therefore, the queue reaches its maximum length at $t = 80$ min, and the maximum queue length is 550 vehicles.



Queuing System Example: D/D/1 (time-varying)

- ***Solution (continued)***

➔ Longest wait time: *The wait time d is the horizontal distance between A-curve and D-curve which can be found by inversing both functions*

$$W = t_D - t_A = \left(30 + \sqrt{10D}\right) - \frac{A}{10}$$

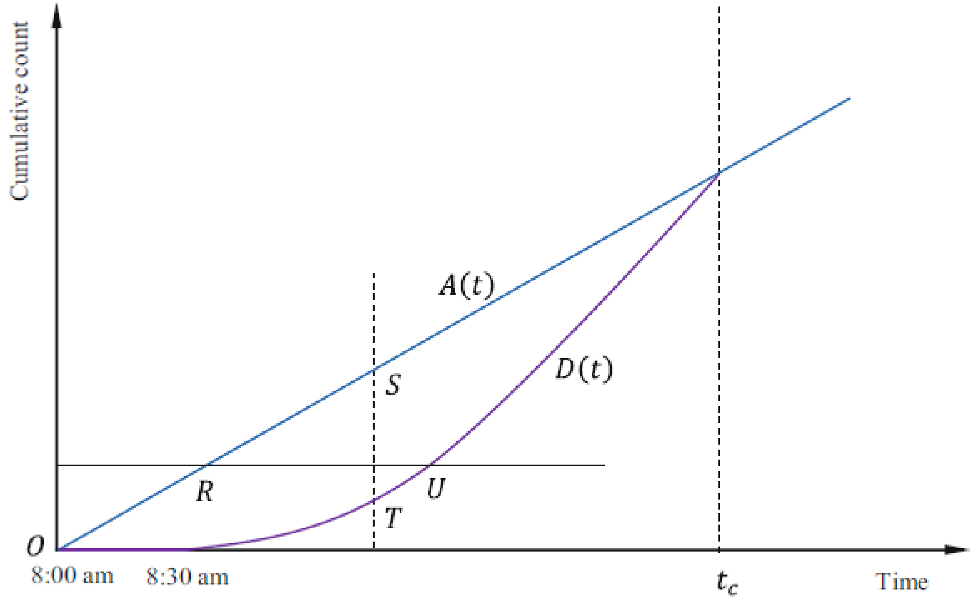
Since we are dealing with the same vehicle: A = D = N:

$$W(N) = \left(30 + \sqrt{10N}\right) - \frac{N}{10}$$

$$\frac{dW}{dN} = \frac{10}{2\sqrt{10N}} - \frac{1}{10} = 0, \quad N = 250$$

$$W(250) = \left(30 + \sqrt{10 \times 250}\right) - \frac{250}{10} = 55$$

The longest wait time is 55 min which is experienced by the vehicle numbered 250



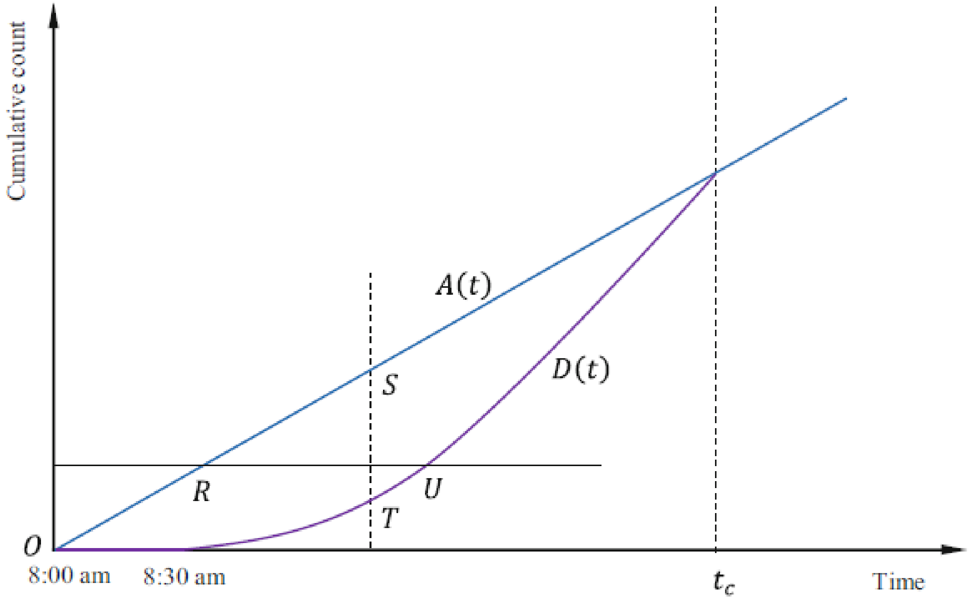
Queuing System Example: D/D/1 (time-varying)

➢ ***Solution (continued)***

➜ **<u>Total delay:</u>** Area bounded by A-curve and D-curve

$$\Gamma = \int_0^t (A(\xi) - D(\xi))d\xi = \int_0^{30} (10\xi - 0)d\xi +$$

$$\int_0^{154.16} \left[10\xi - \left(0.1\xi^2 - 6\xi + 90\right)\right]d\xi \approx 4500 + 50,526 = 55,026 \text{ veh min}$$



➜ **<u>Average delay:</u>**

There will be $N = 10 \times 154.16 = 1541.6$ vehicles arrived at t_c. Therefore, the average delay per vehicle, W, is:

$$W = \frac{\Gamma}{N} = \frac{55,026}{1541.6} \approx 35.7 \text{ min}$$

Queuing System: M/D/1

- If a queuing system has deterministic departure rate μ, but its arrival rate λ follows Poisson distribution or its inter-arrival time h follows exponential distribution, an $M/D/1$ queuing system is obtained.
- The graphical approach only applies to deterministic systems.
- When randomness is involved a statistical approach has to be sought.
- We define traffic intensity ρ as the ratio of arrival rate to departure rate :

$$\rho = \frac{\lambda}{\mu}$$

  - μ is the average departure rate
  - λ is the average arrival rate

- In order for a queuing system to be stable, traffic intensity needs to be less than 1

$$0 < \rho < 1$$

Queuing System: M/D/1

- *Average number of customers in system L*

$$L = \frac{(2-\rho)\rho}{2(1-\rho)} \qquad L = L_Q + \rho$$

- *Average queue length L_Q*

$$L_Q = \frac{\rho^2}{2(1-\rho)}$$

- *Average delay or time spent in system W*

$$W = \frac{2-\rho}{2\mu(1-\rho)} \qquad W = W_Q + \frac{1}{\mu}$$

- *Average waiting time in queue W_Q*

$$W_Q = \frac{\rho}{2\mu(1-\rho)}$$

Queuing System Example: M/D/1

- ***Example***
  Vehicles arrive at a toll booth at an average rate of 2 per minute, and it takes the drivers 20 s to pay toll. Assume that the arrival rate follows Poisson distribution and the departure process is deterministic. Determine average queue length, average delay, average waiting time in queue.

Queuing System Example: M/D/1

- ***Example***
  Vehicles arrive at a toll booth at an average rate of 2 per minute, and it takes the drivers 20 s to pay toll. Assume that the arrival rate follows Poisson distribution and the departure process is deterministic. Determine average queue length, average delay, average waiting time in queue.

- ***Solution***

M/D/1 queuing system

Average arrival rate

$$\lambda = 2 \text{ veh/min}$$

Constant service rate

$$\mu = \frac{60}{20} = 3 \text{ veh/min}$$

Traffic intensity

$$\rho = \frac{2}{3}$$

Average queue length

$$L_Q = \frac{\rho^2}{2(1-\rho)} = \frac{(2/3)^2}{2(1-(2/3))} = \frac{2}{3} \text{ vehicle}$$

Average delay

$$W = \frac{2-\rho}{2\mu(1-\rho)} = \frac{2-(2/3)}{2(3)(1-(2/3))} = \frac{2}{3} \text{ min}$$

Average waiting time in queue

$$W_Q = \frac{\rho}{2\mu(1-\rho)} = \frac{(2/3)}{2(3)(1-(2/3))} = \frac{1}{3} \text{ min}$$

Note that, if the system were D/D/1, there would have been no delay at all since the service rate is faster than the arrival rate.

Queuing System: M/M/1

- A queuing system with Poisson distributed arrival rate (or exponentially distributed inter-arrival time) and Poisson distributed departure rate (or exponentially distributed inter-departure time)

- *Average number of customers in system L*

$$L = \frac{\rho}{1-\rho} \qquad L = L_Q + \rho$$

- *Average queue length L_Q*

$$L_Q = \frac{\rho^2}{1-\rho}$$

- *Average delay or time spent in system W*

$$W = \frac{1}{\mu - \lambda} = \frac{1}{\mu(1-\rho)} \qquad W = W_Q + \frac{1}{\mu}$$

- *Average waiting time in queue W_Q*

$$W_Q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu(1-\rho)}$$

Queuing System Example: M/M/1

- ***Example***
  Vehicles arrive at a toll booth at an average rate of 2 per minute, and it takes the drivers 20 s on average to pay toll. Assume that the arrival rate follows Poisson distribution and the service time is exponentially distributed. Determine average queue length, average delay, average waiting time in queue.

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- ***Solution***

M/M/1 queuing system

Average arrival rate

$$\lambda = 2 \text{ veh/min}$$

Average departure rate

$$\mu = \frac{60}{20} = 3 \text{ veh/min}$$

Traffic intensity

$$\rho = \frac{2}{3}$$

Average queue length

$$L_Q = \frac{\rho^2}{(1-\rho)} = \frac{(2/3)^2}{(1-(2/3))} = \frac{4}{3} \text{ vehicle}$$

Average delay

$$W = \frac{1}{\mu(1-\rho)} = \frac{1}{(3)(1-(2/3))} = 1 \text{ min}$$

Average waiting time in queue

$$W_Q = \frac{\rho}{\mu(1-\rho)} = \frac{(2/3)}{(3)(1-(2/3))} = \frac{2}{3} \text{ min}$$

It is clear that, passing from D/D/1 to M/D/1 to M/M/1, the randomness of the system is increasing, so are queues and delays.

Queuing System: M/M/N

- A more general formulation of the $M/M/1$ queue is the $M/M/N$ queue, where N is the total number of departure channels.
- $M/M/N$ queuing is a reasonable assumption at toll booths on turnpikes or at toll bridges, where there is often more than one departure channel available.
- A parking lot is another example, with N being the number of parking stalls in the lot and the departure rate, μ, being the exponentially distributed times of parking duration.
- $M/M/N$ queuing is also frequently encountered in non-transportation applications such as checkout lines at retail stores, security checks at airports, and so on.
- Unlike the equations for $M/D/1$ and $M/M/1$, which require that the traffic intensity, ρ, be less than 1, the following equations allow ρ to be greater than 1 but apply only when ρ/N (which is called the utilization factor) is less than 1.

Queuing System: M/M/N

$$P_0 = \frac{1}{\sum_{n_c=0}^{N-1} \frac{\rho^{n_c}}{n_c!} + \frac{\rho^N}{N!(1-\rho/N)}}$$

$$P_n = \frac{\rho^n P_0}{n!} \qquad \text{for } n \leq N$$

$$P_n = \frac{\rho^n P_0}{N^{n-N} N!} \qquad \text{for } n \geq N$$

$$P_{n>N} = \frac{P_0 \, \rho^{N+1}}{N! N(1-\rho/N)}$$

where

P_0 = probability of having no vehicles in the system,
P_n = probability of having n vehicles in the system,
$P_{n>N}$ = probability of waiting in a queue (the probability that the number of vehicles in the system is greater than the number of departure channels),
n = number of vehicles in the system,
N = number of departure channels,
n_c = departure channel number, and
ρ = traffic intensity (λ/μ).

Queuing System: M/M/N

$$\overline{Q} = \frac{P_0\, \rho^{N+1}}{N!N}\left[\frac{1}{(1-\rho/N)^2}\right]$$

$$\overline{w} = \frac{\rho + \overline{Q}}{\lambda} - \frac{1}{\mu}$$

$$\overline{t} = \frac{\rho + \overline{Q}}{\lambda}$$

where

$\overline{Q}$ = average length of queue (in vehicles),
$\overline{w}$ = average waiting time in the queue, in unit time per vehicle,
$\overline{t}$ = average time spent in the system, in unit time per vehicle, and
Other terms are as defined previously.

Queuing System Example: M/M/N

- ***Example***
  At an entrance to a toll bridge, four toll booths are open. Vehicles arrive at the bridge at an average rate of 1200 veh/h, and at the booths, drivers take an average of 10 seconds to pay their tolls. Both the arrival and departure headways can be assumed to be exponentially distributed. How would the average queue length, time in the system, and probability of waiting in a queue change if a fifth toll booth were opened?

Queuing System Example: M/M/N

- ***Solution***

M/M/N queuing

We first compute the four-booth case

$\mu = 6\ veh/\min$

$\lambda = 20\ veh/\min$

$\rho = 3.333$

$\rho/N = 0.833$

The probability of having no vehicles in the system with four booths open

$$P_0 = \frac{1}{1 + \frac{3.333}{1!} + \frac{3.333^2}{2!} + \frac{3.333^3}{3!} + \frac{3.333^4}{4!(0.1667)}}$$
$$= 0.0213$$

The average queue length is

$$\bar{Q} = \frac{0.0213(3.333)^5}{4!4}\left[\frac{1}{(0.1667)^2}\right]$$
$$= 3.287 \text{ veh}$$

The average time spent in the system is

$$\bar{t} = \frac{3.333 + 3.287}{20}$$
$$= 0.331 \text{ min/veh}$$

And the probability of having to wait in a queue is:

$$P_{n>N} = \frac{0.0213(3.333)^5}{4!4(0.1667)}$$
$$= 0.548$$

Queuing System Example: M/M/N

- ***Solution (continued)***

With a fifth booth open, the probability of having no vehicles in the system is

$$P_0 = \frac{1}{1 + \frac{3.333}{1!} + \frac{3.333^2}{2!} + \frac{3.333^3}{3!} + \frac{3.333^4}{4!} + \frac{3.333^5}{5!(0.3333)}}$$

$$= 0.0318$$

The average queue length is

$$\bar{Q} = \frac{0.0318(3.333)^6}{5!5}\left[\frac{1}{(0.3333)^2}\right]$$

$$= 0.654 \text{ veh}$$

The average time spent in the system is

$$\bar{t} = \frac{3.333 + 0.654}{20}$$

$$= 0.199 \text{ min/veh}$$

And the probability of having to wait in a queue is:

$$P_{n>N} = \frac{0.0318(3.333)^6}{5!5(0.3333)}$$

$$= 0.218$$

So opening a fifth booth would reduce the average queue length by 2.633 veh (3.287 − 0.654), the average time in the system by 0.132 min/veh (0.331 − 0.199), and the probability of waiting in a queue by 0.330 (0.548 − 0.218).

Queuing System Example: M/M/N

- ***Example***
  A convenience store has four available parking spaces. The owner predicts that the duration of customer shopping (the time that a customer's vehicle will occupy a parking space) is exponentially distributed with a mean of 6 minutes. The owner knows that in the busiest hour customer arrivals are exponentially distributed with a mean arrival rate of 20 customers per hour. What is the probability that a customer will not find an open parking space when arriving at the store?

Queuing System Example: M/M/N

- ***Solution***

Mean arrival time

$$\lambda = 20\, veh/\text{min}$$

Mean departure time

$$\mu = 10\, veh/\text{min}$$

$\rightarrow \rho = 2.0 \quad \rightarrow \quad \rho/N = 0.5$ *(less than 1)*

The probability of having no vehicles in the system with four parking spaces available

$$P_0 = \frac{1}{1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!(0.5)}}$$

$$= 0.1304$$

The probability of not finding an open parking space upon arrival is

$$P_{n>N} = \frac{0.1304(2)^5}{4!4(0.5)}$$

$$= \underline{\underline{0.087}}$$

Traffic Analysis at Highway Bottlenecks

- Some of the most severe congestion problems occur at highway bottlenecks, which are defined as a portion of highway with a lower capacity than the incoming section of highway.
- There are two general types of traffic bottlenecks
  - Recurring
    - ✓ Where the highway itself limits capacity by physical reduction in the number of lanes.
    - ✓ Recurring traffic flows that exceed the vehicle capacity of the highway in the bottleneck area
  - Incident induced
    - ✓ A result of vehicle breakdowns or accidents
    - ✓ The capacity may change over time. For example, an accident may initially stop traffic flow completely, but as the wreckage is cleared, partial capacity may be provided for a period of time before full capacity is eventually restored.
- The most common approach to analyzing traffic congestion at bottlenecks is to assume $D/D/1$ queuing

D/D/1 Queuing: Highway Bottleneck Application

- ***Example***
  An incident occurs on a freeway that has a capacity in the northbound direction, before the incident, of $4000\ veh/h$ and a constant flow of $2900\ veh/h$ during the morning commute (no adjustments to traffic flow result from the incident). At $8{:}\,00\ A.M$. a traffic accident closes the freeway to all traffic. At $8{:}\,12\ A.M$. the freeway is partially opened with a capacity of $2000\ veh/h$. Finally, the wreckage is removed, and the freeway is restored to full capacity ($4000\ veh/h$) at $8{:}\,31\ A.M$. Assume $D/D/1$ queuing to determine time of queue dissipation, longest queue length, total delay, average delay per vehicle, and longest wait of any vehicle (assuming $FIFO$).

D/D/1 Queuing: Highway Bottleneck Application

➢ ***Solution***

➜ $\mu = \dfrac{4000 \text{ veh/h}}{60 \text{ min/h}} = 66.67 \text{ veh/min}$ μ : Full-capacity departure rate

➜ $\mu_r = \dfrac{2000 \text{ veh/h}}{60 \text{ min/h}} = 33.33 \text{ veh/min}$ μ_r : Restrictive partial-capacity departure rate

➜ $\lambda = \dfrac{2900 \text{ veh/h}}{60 \text{ min/h}} = 48.33 \text{ veh/min}$

➜ The arrival rate is constant over the entire time period, and the total number of vehicles is equal to λt, where t is the number of minutes after 8:00 A.M.

➜ The total number of departing vehicles is

$$\begin{array}{ll} 0 & \text{for } t \le 12 \text{ min} \\ \mu_r (t-12) & \text{for } 12 \text{ min} < t \le 31 \text{ min} \\ 633.33 + \mu(t-31) & \text{for } t > 31 \text{ min} \end{array}$$

$[33\tfrac{1}{3}(31 - 12)] \rightarrow 633.33$

D/D/1 Queuing: Highway Bottleneck Application

- ***Solution (continued)***

➜ *These arrival and departure rates can be represented graphically*

➜ *The queue will dissipate at the intersection point of the arrival and departure curves* $\lambda t = 633.33 + \mu(t - 31)$ or $t = \underline{78.16 \text{ min}}$ (just after 9:18 A.M.)

➜ *The longest queue occurs at 8:31 a.m.*

$$
\begin{aligned}
Q_{max} &= \lambda t - \mu_r (t - 12) \\
&= 48.33(31) - 33.33(19) \\
&= \underline{\underline{865 \text{ veh}}}
\end{aligned}
$$

➜ *At this time a total of* 3777.5 *vehicles* (48.33×78.16) *will have arrived and departed*

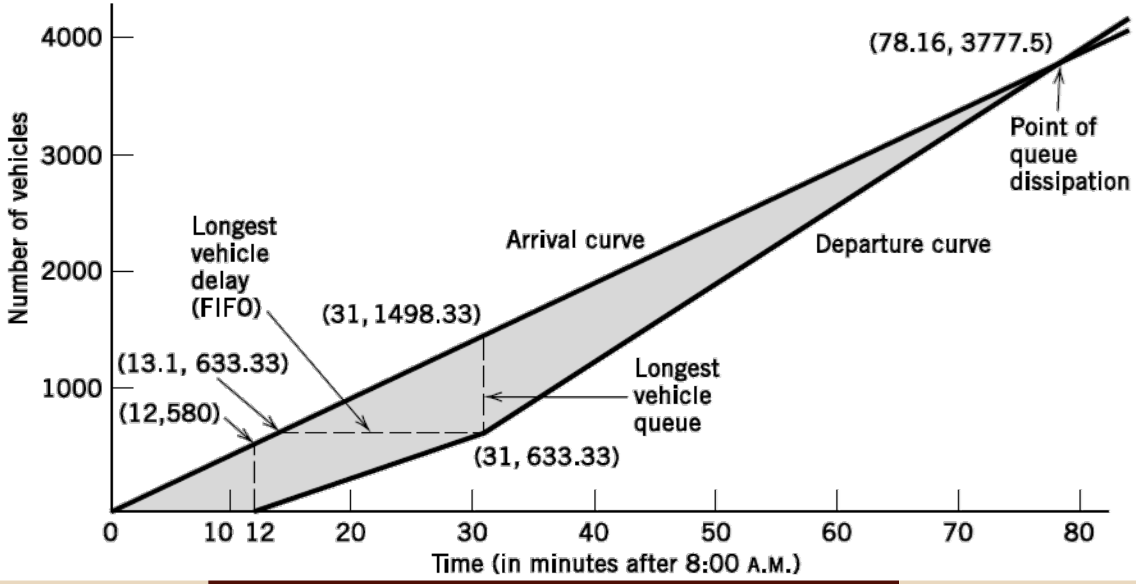
➜ *Total vehicle delay is:*

$$
\begin{aligned}
D_t &= \frac{1}{2}(12)(580) + \frac{1}{2}(580 + 1498.33)(19) - \frac{1}{2}(19)(633.33) \\
&\quad + \frac{1}{2}(1498.33 - 633.33)(78.16 - 31) \\
&= 37{,}604.2 \text{ veh-min}
\end{aligned}
$$

➜ *The average delay per vehicle is 9.95 min.*

➜ *The longest wait of any vehicle will be experienced by the 633.33rd vehicle.*

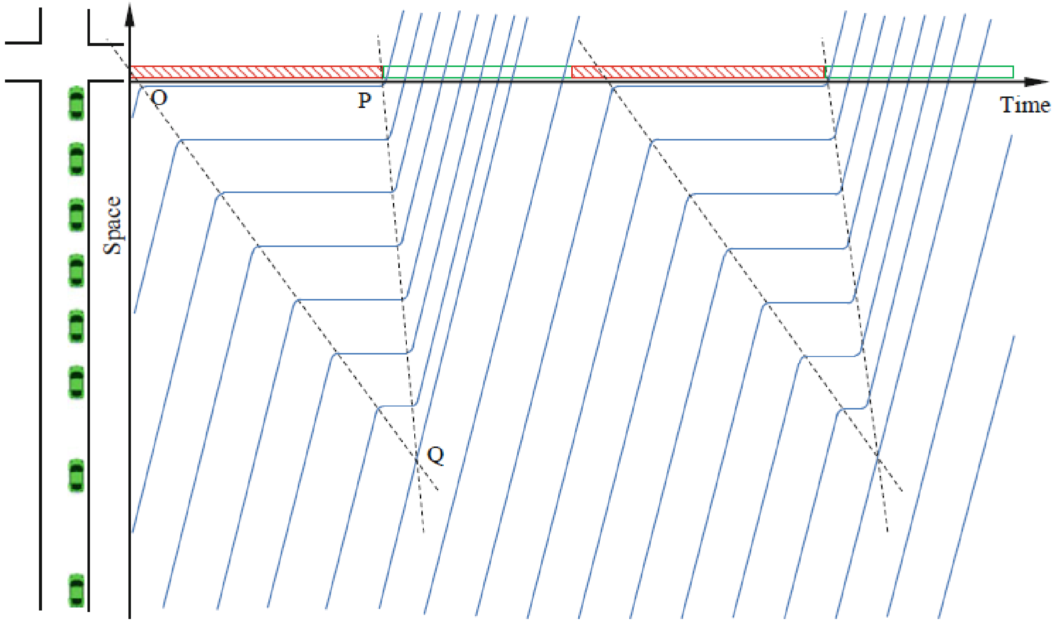
➜ *This vehicle will arrive 13.1 minutes after 8:00 A.M. and will depart at 8:31 A.M., being delayed a total of 17.9 min.*



Queuing at Signalized Intersections

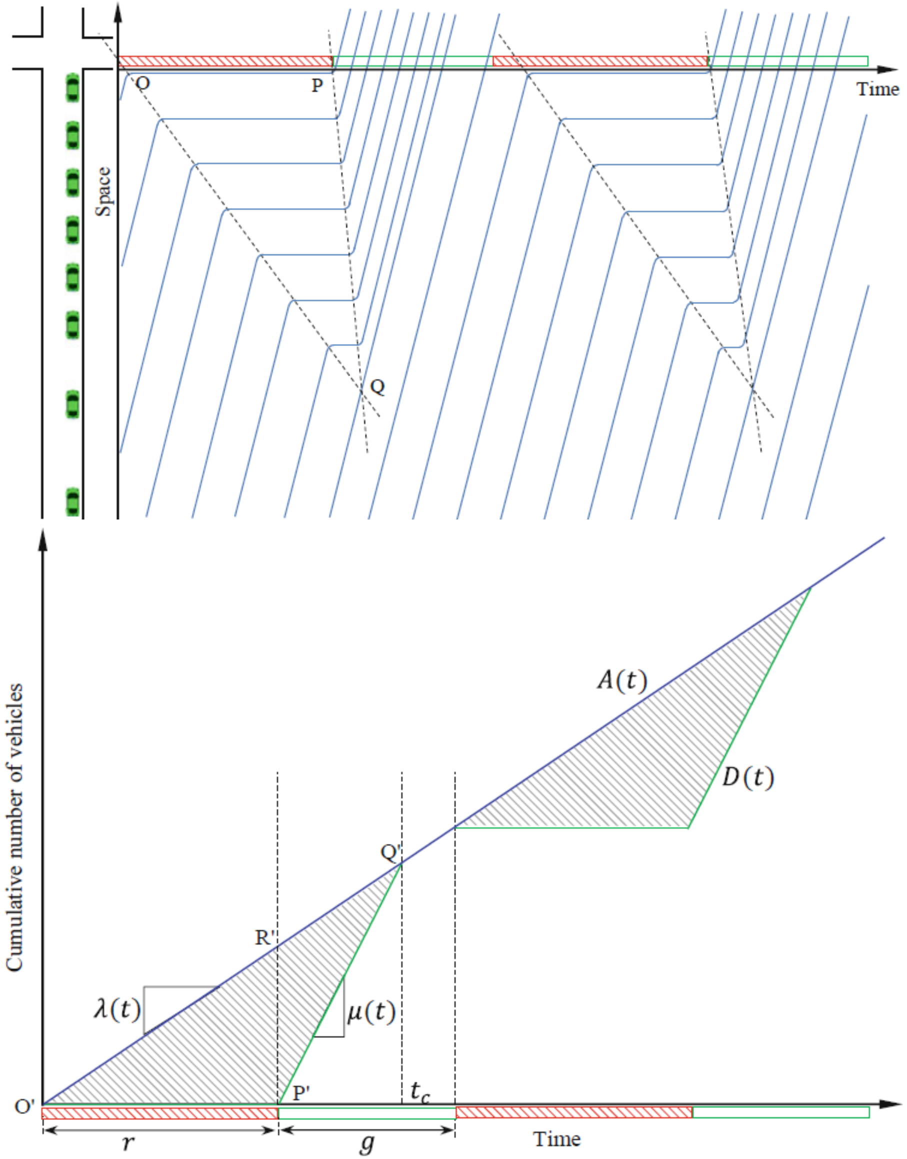
- Time-space diagram to illustrate traffic operation at an approach of a signalized intersection.
- Each curve represents a vehicle trajectory.
- Traffic signal for this approach is represented by a bar consisting of alternating effective red and effective green.
- Vehicle arriving at this approach on effective red will stop and form a queue.
- Consequently, a ***shock wave*** (line OQ) is generated indicating the time space location of the tail of the queue.

- When effective green comes, vehicles in the queue begin to discharge, forming another shock wave (line PQ) indicating the time space location of the head of the queue.
- As the two shock waves meet (at point Q), the queue is dissipated, after which vehicles arrive and depart without stopping.
- The smooth operation continues until the next effective red comes, after which another queue begins to build up and the above processes repeat for another cycle.



Queuing at Signalized Intersections

- The time-space diagram can be transformed to an equivalent time-cumulative number of vehicles diagram.
- Points O, P, and Q correspond to O',P', andQ', respectively.
- Line $O'Q'$ represents cumulative arrival curve $A(t)$ whose slope is the arrival rate $\lambda(t)$.
- Curve $O'P'Q'$ represents cumulative departure curve $D(t)$ whose slope is the departure rate $\mu(t)$.
- Effective green g,
- Effective red r,
- Arrival rate λ,
- Departure rate μ



Queuing at Signalized Intersections

- Traffic operation in one cycle can be analyzed as follows:

Cumulative arrival curve $A(t)$: $$A(t) = \int_0^t \lambda d\xi = \lambda t$$

Cumulative Departure curve $D(t)$:

$$D(t) = \int_0^t \mu(\xi) d\xi = \begin{cases} \int_0^t 0 d\xi = 0 & \text{for } 0 \le t < r \\ \int_0^g 0 d\xi + \int_g^t \mu d\xi = \mu(t-r) & \text{for } r \le t < r+g \end{cases}$$

Queue clearance time t_c:

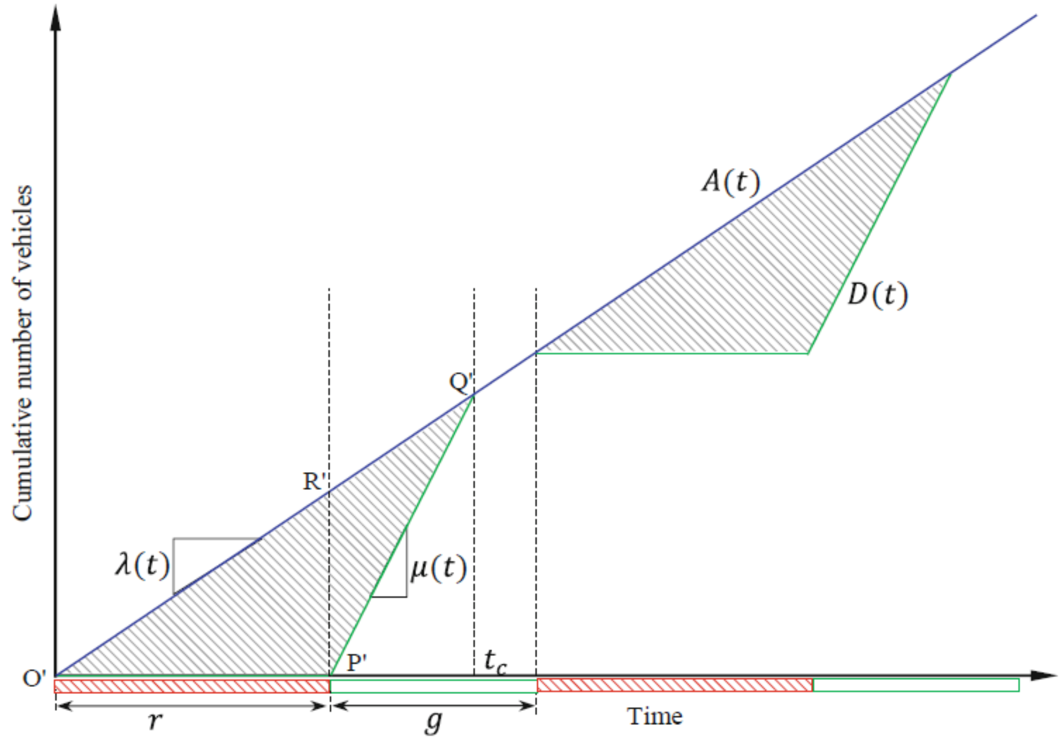
$$\lambda t_c = \mu(t_c - r), \quad t_c = \frac{\mu r}{\mu - \lambda} = \frac{r}{1-\rho}$$

Longest queue L_m:

$$L_m = \lambda r$$

Longest wait time W_m:

$$W_m = r$$



Queuing at Signalized Intersections

➢ Traffic operation in one cycle can be analyzed as follows:

Total delay Γ: *The Area bounded by A-curve and D-curve* $\Gamma = \int_0^t (A(\xi) - D(\xi))d\xi = \int_0^r (\lambda\xi - 0)d\xi + \int_r^{t_c} [\lambda\xi - \mu(\xi - r)]d\xi = \frac{1}{2}\lambda r t_c$

Average delay W: *The total number of vehicles arrived in a cycle* $N = \lambda C$ $W = \frac{\frac{1}{2}\lambda r t_c}{\lambda C} = \frac{r t_c}{2C}$

Average queue length L: *Total delay divided by cycle length*

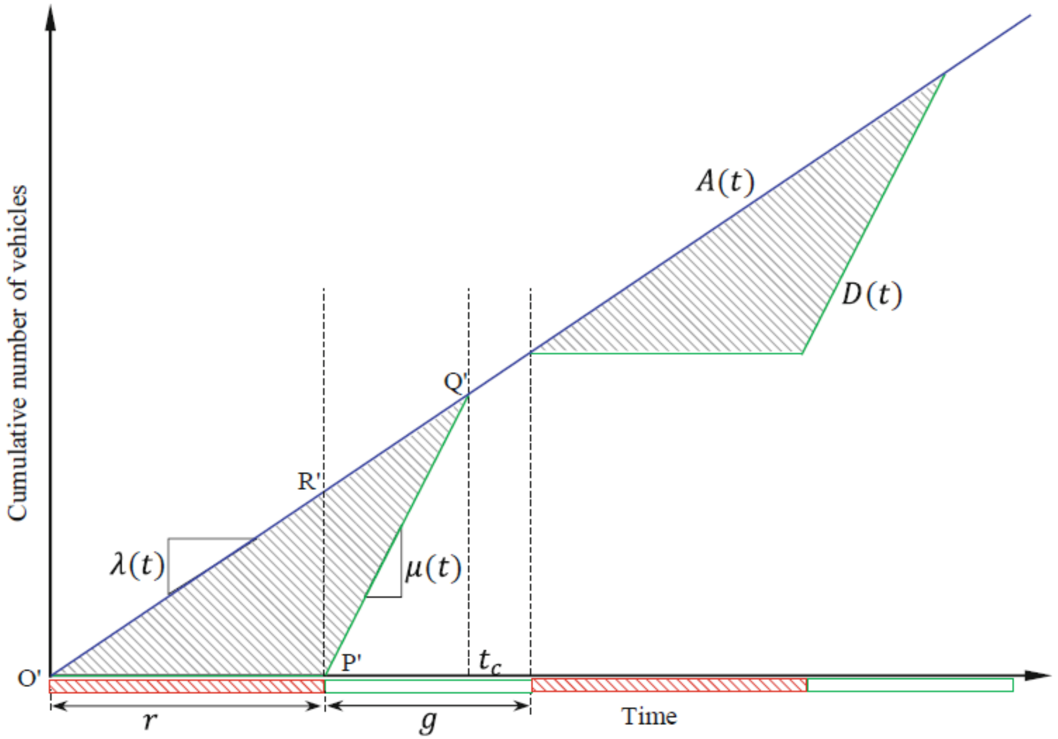
$$L = \frac{\frac{1}{2}\lambda r t_c}{C} = \frac{\lambda r t_c}{2C}$$

Proportion of the cycle with a queue P_Q: $P_Q = \frac{t_c}{C} = \frac{t_c}{g + r}$

t_c *out of a cycle* $C = g + r$

Proportion of vehicles having to stop P_S:

The total number of vehicles in a cycle is λC*, while the number of stopped vehicles is* $\lambda\ tc$ ➜ $P_S = \frac{t_c}{C} = \frac{t_c}{g + r} = P_Q$



Queuing at Signalized Intersections

- Note that the above analysis is based on two implicit assumptions:
    1. There is no initial queue
    2. The queue dissipates within the cycle, i.e., no queue spills onto the next cycle.
- If either of these assumptions is not met, the above equations do not apply.
- Instead, one needs to manually analyze the queuing system using knowledge of $D/D/1$ queuing.
- To check the second assumption:
    - Cumulative arrival $A(t)$ at the end of the first cycle: $A(t) = \lambda C$
    - Cumulative departure $D(t)$ at the end of the first cycle assuming there is enough vehicles to discharge: $D(t) = \mu g$
    - The second assumption is met when: $D(t) \geq A(t)$ or $\mu g \geq \lambda C$

Queuing at Signalized Intersections

- ***Example***
  A stream of traffic flow of 720 veh/h arrives at an approach of a signalized intersection which operates a two-phase pre-timed signal. The signal cycle length is 60 s, and the effective green on the above approach is 30 s. Field experiment shows that saturation headway on this approach is 2 s per vehicle. Assume that this approach operates as a D/D/1 queuing system, analyze traffic operation at this approach and provide statistics to quantify traffic operation.

Queuing at Signalized Intersections

- ***Solution***

$\lambda = \frac{720}{3600} = \frac{1}{5}$ veh/s

$\mu = \frac{1}{2}$ veh/s

$\rightarrow \quad \rho = \frac{1/5}{1/2} = \frac{2}{5}$

$g = 30$ s $\qquad r = C - g = 30$ s

Check if the two implicit assumptions are met:

- *There is no initial queue, so the first implicit assumption is met;*
- *At the end of the first cycle:* $A(t) = \lambda C = 1/5(60) = 12$, $D(t) = \mu g = 0.5 * 30 = 15$. $D(t) \geq A(t)$ *holds, so the second implicit assumption is met.*

Queue clearance time: $t_c = \frac{r}{1-\rho} = \frac{30}{1-\frac{2}{5}} = 50$ s

Longest queue: $L_m = \lambda r = \frac{1}{5}(30) = 6$ veh

Longest wait time: $W_m = r = 30$ s

Total delay: $\Gamma = \frac{1}{2}\lambda r t_c = \frac{1}{2}\left(\frac{1}{5}\right)(30)(50) = 150$ veh/s

Average delay: $W = \frac{r t_c}{2C} = \frac{30(50)}{2(60)} = 12.5$ s

Average queue length: $L = \frac{\lambda r t_c}{2C} = \frac{\frac{1}{5}(30)(50)}{2(60)} = 2.5$ veh

Proportion of the cycle with a queue: $P_Q = \frac{t_c}{C} = \frac{50}{60} = \frac{5}{6}$

Proportion of vehicles having to stop: $P_S = P_Q = \frac{5}{6}$

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Thank you
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