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Optimized Financial Systems Helps Customers Meet Their Personal Finance Goals with Optimization

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We consider the retirement planning question that a single taxpayer or couple faces when determining an amount to save for retirement and the type of retirement account in which to allocate the assets. Additionally, taxpayers can choose to convert savings from one account type, such as a traditional individual retirement arrangement (IRA), into another, such as a Roth IRA; all such choices have both immediate and long-term tax consequences. We give the rationale for applying stochastic optimization in this setting and explain how such an optimization model can be constructed as a multiple-stage stochastic linear or mixed-integer program. Optimized Financial Systems implements such a model and uses it to advise clients; we demonstrate the approach's benefits by examining its recommendations both for test data and in a case study based on an actual customer.

Keywords: personal finance; financial planning; stochastic optimization.
History: This paper was refereed. Published online in *Articles in Advance* June 2, 2016.

Individual retirement arrangements (IRAs) were introduced in 1974 to encourage individual savings. The Internal Revenue Service in publication 590-A (IRS 2014) defines IRAs as individual retirement arrangements; this refers to the broad class of account types an individual may have. A person may own one or more IRAs. Traditional IRA contributions allow a taxpayer to reduce gross income and the resulting taxes. Traditional IRAs are known as front-loaded IRAs; the money grows and is tax deferred until the taxpayer withdraws it. These withdrawals are known as distributions and are taxed as ordinary income. Since 1974, Congress has raised annual IRA contribution limits and modified eligibility. Today, an individual with earned income may make a contribution of up to $5,500 per year if under 50, and $6,500 if 50 or over. If the individual is covered by an employer-sponsored retirement plan and has income above $59,000 (single) or $95,000 (married, filing jointly), the deductibility of the contributions is reduced and the IRA acquires a basis. The basis of an IRA is the

total contributions to the IRA that were not deducted from current income; it reduces the future taxability of distributions and conversions from an IRA. For example, if an IRA has a basis of 10 percent, only 90 percent of all distributions and conversions from that account will be taxable. IRA distributions may begin penalty free at age $59\frac{1}{2}$ and must commence the year the account holder turns $70\frac{1}{2}$ (Hungerford and Gravelle 2010).

Roth IRAs, which are back-loaded qualified retirement plans, were introduced in the Taxpayer Relief Act of 1997. In a Roth IRA, contributions are included as ordinary income but distributions are not taxable and the contents of a Roth IRA need never be distributed to the original owner. An individual with an adjusted gross income higher than a given limit is not allowed to contribute to a Roth IRA. Starting in 2006, employers were also able to offer the Roth version of the 401(k) plan, which does not have income restrictions. Moreover, converting IRA assets to Roth IRA assets is possible; initially, this was allowed only

for individuals with modified adjusted gross incomes below $100,000; however, Congress removed this cap in 2009 beginning with the 2010 tax year. These legislative changes gave taxpayers new options for personal financial planning, such as determining whether to convert their IRA assets to a Roth IRA in 2010 (Hoyt 2010, Kaplan 2009), or determining how a Roth IRA fits into a diversified tax strategy (Ashby et al. 2010). In addition to determining which investments to have in their portfolios, U.S. investors also need to determine the account types for their investments (Dammon et al. 2005); this is an asset-location problem.

Objective

Our objective in this paper is to demonstrate how an optimization model that leverages actuarial information can provide detailed personal finance plans to help investors determine the types of accounts in which to place their investments. In particular, we show that resulting plans are superior to the alternative of exclusively choosing one account type and entirely converting or not converting existing IRA and 401(k) accounts to their Roth counterparts. When evaluating a Roth conversion, an optimization model can simultaneously consider multiple mortality outcomes, marginal tax rates, conversion amounts, and any other relevant information. For example, a conversion or a distribution may be large enough to touch several marginal rates; therefore, the ability to convert smaller amounts over time results in increased wealth for the individual. Furthermore, the majority of constraints in such a model are linear, yielding a straightforward, computationally attractive formulation. The model determines the following.

1. The amount to convert each year.

2. The amount to contribute to each IRA, Roth IRA, 401(k) or 403(b), and Roth 401(k) or Roth 403(b) account each year.

3. The source of distributions.

4. The amount to spend each year (up to a budgeted input).

We organized the remainder of the paper in the following manner. The *Literature Review* outlines research in asset allocation and location, and the benefits of using IRAs and Roth IRAs. The *Approach* section explains the uses of the optimization mode for

personal financial planning and discusses its objectives. The *Experiments* section summarizes detailed experiments that compare the model's solutions to other conversion strategies. In the *Case Study* section, we present a case study using real-world data. Finally, we finish with a *Conclusions* section. The *Appendix* contains the mathematical formulation of the model and an explanation of its parameters, decision variables, objective functions, and constraints.

Literature Review

Since the introduction of IRAs and tax-deferred accounts (TDAs), several authors have written about both asset allocation and asset location. The asset-allocation problem determines the percentage of assets that an overall portfolio should have, whereas the asset-location problem determines the account (location) in which each asset belongs. Shoven and Sialm (2004) address the asset-allocation and asset-location problems with the goal of maximizing expected utility. The conventional wisdom was to place assets that might be subject to higher tax rates in a TDA; they recommend switching to tax-free municipal bonds in the taxable account and moving stocks to the TDA once stocks reach 52 percent of a portfolio's allocation. Poterba et al. (2000) also contradict the notion that bonds should be held in TDAs. Using data from 1962 to 1998, they compare two asset-location strategies for retirement savers who invest in equity mutual funds. They offer asset-location rules based on effective tax rates on equities, bonds, and the implicit tax rate on tax-exempt bonds. When evaluating equity index funds against actively managed, tax-inefficient mutual funds, equity index funds outperform the actively managed funds. Only when using more tax-efficient index funds does a defer-bonds-first strategy prevail; Dammon et al. (2005) come to similar conclusions. They also recommend increasing the percentage of bonds as the percentage of assets in a TDA increases. These papers were written when dividends were taxed at ordinary income rates. The proper asset location can make even more of a difference today because qualified dividends are taxed at long-term capital gains rates.

Despite the benefits of proper asset location, Bergstresser and Poterba (2004) find that many investors

are not making the most of their TDAs. They analyze asset allocation and location among TIAA-CREF plan participants and find similar allocations in both tax-deferred and taxable accounts. They conclude that savers would be better off moving their highly taxable investments to TDAs. If available as an option, they suggest using tax-free bonds in taxable accounts and equities in the tax-deferred accounts. This is relevant to a large part of the U.S. population; Poterba et al. (2011a) note that 52.2 percent of households have assets in personal retirement accounts (PRAs). Poterba et al. (2011b) also note that 33 years ago, retirees were usually covered by defined-benefit plans, whereas today PRAs are the primary form of retirement savings for individuals. As a result, today's retirees are more concerned with outliving their assets, and the authors conclude that there is a general pattern of conserving assets in TDAs. In general, Bernheim et al. (2000) assert that savings rates must increase, especially for people with higher incomes who wish to maintain their lifestyles in retirement, but they do not include the effects of tax-advantaged accounts in these calculations. Davidoff et al. (2003) find the choice to annuitize savings is not always clear, especially when potential annuitants are unable to make trade-offs between different multiple periods and states of wealth. Another issue with annuitization calculations is the assumption of no bequest motive. Milevsky et al. (2006) advance the annuity discussion; they add variable returns and the objective to avoid financial ruin, defined as running out of money before dying. They conclude that investments in risky assets increase with wealth.

Gokhale et al. (2001) find that low- and middle-income households raise their lifetime taxes and lower their lifetime expenditures by investing in 401(k) accounts. They did not include the Roth option in these calculations, but note this option could be of value. For the wealthy, 401(k) accounts offer substantial tax savings and wealth gains. For low- and middle-income households, investing in Roth accounts affords much greater lifetime tax benefits. The reduction in lifetime expenditures is the result of the increased taxability of social security benefits.

Several articles in the literature make recommendations about the best use of TDAs and Roth accounts. Burman et al. (2001) determine the applicability of

Roth accounts for many taxpayers. They examine actual returns of a set of taxpayers for tax years 1982 and 1995, using both fixed tax regimes and the actual tax regimes. The results show the percentage of people by income and age group that would benefit from Roth IRA contributions over IRA contributions. They note that "even if tax rates fall significantly in retirement, taxpayers may still prefer to use back-loaded savings vehicles" (Burman et al. 2001, p. 20), which goes against conventional wisdom. Dammon (2009) analyzes situations in which a person would choose to make a conversion and when the conversion would be beneficial. This work compares the marginal rates at the time of conversion, the number of years until withdrawal, the marginal rate at distribution, whether the taxes are paid from within the IRA, whether an early withdrawal penalty is also paid from the conversion, or whether assets from a taxable account are used to pay the taxes. Burman et al. (2001) use historical tax records and hindsight to make the conversion determination based on simulated results, while Dammon (2009) derive formulas to determine when it would be advantageous to make Roth conversions. Both come to the same conclusion: A Roth conversion may be a good strategy even if the tax rate were to decline between the time of contribution and the time of distribution.

Poterba et al. (2011a) analyze wealth and retirement assets; they note that defined-benefit pension plans and social security payments are "the two most common annuitized income streams" (p. 95). Other household wealth is in the form of home equity, taxable accounts holding stocks and bonds, and retirement accounts such as IRAs and 401(k)s. However, the value of the taxable account is less than $20,000 for 55 percent of households. The authors find that only 47 percent of households aged between 65 and 69 had enough assets to increase their annuity income by more than $5,000 per year. Although early planning is beneficial, we found that retirees can still make changes to their asset-location and distribution strategies to make the most of the assets they have.

The literature includes studies that make recommendations on whether Roth contributions would be more profitable than IRA contributions, and whether a Roth conversion will be profitable based on current and expected tax rates. To our knowledge, however,

no study attempts to determine the exact quantity of a conversion. In addition, the literature includes little work concerning asset portfolios that include taxable accounts, IRAs, and Roth IRAs. By applying optimization to Roth conversions, the model that Optimized Financial Systems (OFS) developed can help an investor to incrementally plan Roth conversions over a lifetime, expand the pool of beneficiaries from a Roth conversion, and encourage individual savings by taking advantage of the combined benefits of Roth and conventional IRAs.

Approach

A traditional approach to most personal finance and tax plans is to make assumptions about current and future tax rates and determine whether a specific conversion is advisable. A common refrain is that if a client will have a lower tax rate in retirement, then that client will not benefit by converting now (Kaplan 2009, VanZante et al. 2005). These recommendations endure despite evidence based on actual returns (Burman et al. 2001) and formulas that give the minimum future tax rate at which a conversion is profitable (Dammon 2009).

Previous analyses, however, show only the benefits of conversion in isolation, making the calculations using one conversion-contribution year and one distribution year. Making conversion decisions in isolation has a drawback. Although a conversion may be profitable as an isolated decision, it may not be the optimal decision. Table 1 shows an example of a conversion that takes place over one or two years. The client wishing to do the conversion has $200,000 in IRA assets and $200,000 in a separate taxable account, which could be used to pay taxes. The assumed rate of return is five percent. The client's tax rates are such that the first $50,000 of the conversion incurs a 25 percent marginal tax rate and the remainder a 28 percent marginal tax rate. Suppose the distributions will take place far enough into the future that either conversion is profitable. In the one-year conversion, all $200,000 of the IRA is converted. The taxes on this conversion are $54,500 (25 percent of $50,000 + 28 percent of $150,000); at the end of the second year, the taxable account is worth $152,775 (1.05 × ($200,000 − $54,500)). Under a two-year conversion,

	One-year conversion ($)	Two-year conversion ($)
Conversion, year 1	200,000	100,000
Tax on conversion, year 1	54,500	26,500
Taxable assets, year 1	145,500	173,500
Roth IRA, year 1	200,000	100,000
Conversion, year 2	—	105,000
Tax on conversion, year 2	—	27,900
Taxable assets, year 2	152,775	154,275
Roth IRA, year 2	210,000	210,000

Table 1: Optimizing conversions can increase a portfolio's value. In this example, by converting over two years instead of one year, the taxable assets increase by $1,500 and the total amount in the Roth IRA remains the same.

$100,000 is converted in the first year, and $105,000 is converted in the second year, because the IRA assets grow from one year to the next. In the first year, the taxes are $26,500 (25 percent of $50,000 + 28 percent of $50,000). In the second year, taxes are $27,900 (25 percent of $50,000 + 28 percent of $50,000). The total taxable assets left in the second year are $154,275 (1.05 × ($200,000 − $26,500) − $27,900). The amount in the Roth account is still the same after the second conversion, but the amount of taxable assets has increased by $1,500. Wealth has increased by $1,500 by deciding to convert over two years.

The example shows how converting over a two-year period increases wealth over a single-year conversion. Spreading the conversion over three, four, or more years is also possible; the results improve each time. If extended over a sufficiently long period, the growth of the IRA assets over time makes the taxes on more conversions at a lower rate outweigh the cost of converting at a higher rate and losing the return on the taxable assets. The number of variables to manage and the trade-offs in taxes and asset growth, make this problem difficult to solve manually or heuristically.

An additional difficulty is how a financial planning model considers mortality; people do not live to their exact life expectancy. A person's age at death is an unknown value that follows a probability distribution. By including a mortality probability distribution, such as the actuarial life table in U.S. Social Security Administration (2007), in an optimization model, it is possible to develop a savings, conversion, and spending plan that is weighted toward the most likely mortality scenarios. A stochastic linear program can keep

track of financial decisions under multiple scenarios and create a robust plan that considers all mortality combinations and their probabilities.

Rockafellar and Uryasev (2000, 2002), and Yang (2010) provide examples of stochastic linear programs for risk management within the finance industry; stochastic linear programming in general has many applications (Sen and Higle 1999). To our knowledge, no one has applied actuarial data and stochastic programming techniques to personal finance and tax planning. Using optimization techniques, we are able to create an individualized plan for spending, retirement contributions, conversions, and distributions. The result is a solution that minimizes the risk of a retiree outliving assets, and maximizes the estate's net present value (NPV).

The model uses an age combination matrix with all the possible mortality combinations. Table 2 shows an example for a couple aged 55 and 50. In the first year, the base year, both members of the client couple are alive. In the next year, one of four possibilities may take place.

1. Both members survive.
2. Both members do not survive.
3. The wife survives and the husband does not survive.
4. The husband survives and the wife does not survive.

Based on the probabilities of these four events each year, the model optimizes a series of two objective functions. The first objective minimizes a probability-adjusted budgetary shortfall over the client's lifespan. The goal is to plan spending so it falls under the client's target budget as much as possible. The second objective uses the resulting budget from the shortfall objective optimization and maximizes the estate's after-tax NPV. This sequential optimization manages the trade-off between two competing interests: the client would like to maximize the estate's value, but would also like to maintain a certain standard of living.

The model uses multiple recourse, where the decisions in one period are affected by the mortality probabilities in that period and downstream periods. The result is a matrix of all age combinations with the upper-left age combination as the starting point with

55, 50	55, 51	55, 52	55, …	55, n
56, 50	**56, 51**	56, 52	56, …	56, n
57, 50	57, 51	**57, 52**	57, …	57, n
⋮, 50	⋮, 51	⋮, 52		…, n
m, 50	m, 51	m, 52	m, …	**m, n**

Table 2: The diagonal, in bold, represents age combinations with no mortality, beginning with the present combination in the top left. The branches going to the right or down from the diagonal represent an alternative where one spouse has died.

the current ages of the couple. For discussion purposes, the value on the left is the husband's age and the value on the right is the wife's age. The diagonal represents the progression of time where no mortality exists. The off-diagonals represent age combinations where there has been single mortality. On the left side of the diagonal (and going down) are the scenarios where the husband outlives the wife. Scenarios where the wife outlives the husband are on the right side of the diagonal, going across.

Many variables in the model derive their values from the values in the previous year. For example, in Table 2, the dividends and interest received at age combinations $(57, 51)$, $(57, 52)$, and $(56, 52)$ depend on the assets at age combination $(56, 51)$. Decisions that take place in $(56, 51)$ will affect decisions in the three downstream age combinations. We use the probabilities in the objective functions to place more weight on events that are more likely to take place. By modeling decisions for all age combinations that have a greater than 0.0001 chance of taking place, the model ensures a robust solution that will be applicable under a wide range of mortality results.

Objective Functions

Customers have two competing goals. One is to maximize their estate's value; the second is to be able to spend enough each year to maintain a desired lifestyle. To consider both, OFS sequentially optimizes two objectives for each client. The first objective function minimizes shortfall, which is based upon a budget, and the second maximizes the estate's NPV. OFS first minimizes the shortfall, or negative deviation from the customer's desired spending level. This optimal shortfall establishes the maximum feasible budget under the assumptions given by the customer. Once OFS establishes an optimal shortfall, the next

step is to maximize the estate's expected NPV, with the added constraint of keeping the budgetary shortfall at or below the value established in the first optimization. We discuss the two sequential optimizations here in qualitative terms; we include a mathematical formulation of a basic version of the optimization model in the appendix, detailing parameters, variables, and typical constraints.

The rationale for sequentially optimizing the two objectives stems from customer feedback. When we only maximized NPV subject to a customer-specified budget, customers either wanted to know how much more they could spend without outliving their retirement savings, or they were overly optimistic in their budgeting. In the latter case, we reran the shortfall minimization several times, increasing or decreasing the budget, until we got slightly positive shortfall values. In finding a minimum shortfall, we determine an appealing and still-feasible budget constraint, which we use when maximizing the NPV.

Shortfall. Shortfall is defined at every age combination as the difference between the after-tax budget and the amount spent at that age combination, when this difference is positive. If a budget is $100 and the client is able to spend only $75, the shortfall is $25. The goal is to ensure that the customers spend such that their spending is as close as possible to their target over their entire lifetime, weighed by the probability they survive to spend the money. Whenever the couple is unable to spend up to its budget, a shortfall occurs. In the objective function, this shortfall value is multiplied by the probability that this age combination will take place, thereby weighing present consumption more than the future.

At every age combination, the probability at present (55, 50 in the Table 2 example) is 100 percent. The probability associated with the next year (56, 51 in Table 2) is the probability that the husband survives multiplied by the probability that the wife survives, which is 98.88 percent if we use the actuarial table in U.S. Social Security Administration (2007). Three other possibilities exist for the next year. The probability for the off-diagonals are the probabilities that the wife dies and the husband survives (56, 50), 0.32 percent, and the probability that the husband dies and the wife survives (55, 51), 0.79 percent. Although these probabilities are based on

government-calculated actuarial data, more accurate actuarial data can be substituted, if available. Each year, the probability of mortality increases; in the model, a shortfall of $10,000 in the base age combination is much more costly than a shortfall of $100,000 40 years into the future at age combination (95, 90), when the probability of still being alive is much less than 10 percent. The model considers all age combinations where the probability of surviving to that age is greater than 0.0001. In a situation such as the example in Table 2, this still accounts for 99.50 percent of the possible age combinations at death.

Net Present Value. The second objective is the estate's expected after-tax NPV. This is the value of the assets in IRAs, Roth IRAs, and taxable accounts. The value of the assets at each age combination is discounted to the present and further weighted by the probability of dying at that age combination. The discount factor is calculated using the expected returns on assets. This is the sum of the growth, interest, and dividend rates.

Finally, the asset values are also adjusted based on their tax treatment. In a taxable account, dividends and interest are taxed at their respective rates. Inherited IRAs and inherited Roth IRAs have mandatory distributions. These distributions are over the expected lifetime of the inheritor. The younger an inheritor, the smaller the distributions may be, and the longer the assets accumulate as tax deferred. This is known as the stretch. In an inherited IRA, the distributions are taxed and the assets grow untaxed until they are distributed. In an inherited Roth IRA, the distributions are untaxed and the account assets grow untaxed. In a taxable account, the interest and dividends are taxed, and capital gains taxes are not due unless some of the assets are sold. After a defined period, all the assets in the inherited IRA or inherited Roth IRA account will have been withdrawn and moved into a taxable account. Inherited IRAs are tax time bombs (Slott 2012); that is, IRAs have required taxable distributions and, if inherited as part of an estate, they may incur both income tax and estate tax. Our model reflects this.

Experiments

To examine how the optimization model plans a Roth conversion strategy, we ran over 400,000 test cases for

sample couples, where we varied the husband's starting age to be between 30 and 70 in five-year increments; we chose the wife's age to be the husband's age or five years younger. The initial incomes for each spouse were $0, $50,000, $100,000, $200,000, $500,000, or $1,000,000. Initial taxable assets were $0, $100,000, $200,000, $500,000, $1,000,000, and $2,000,000. Initial IRA assets for each spouse were $0, $100,000, $200,000, $500,000, $1,000,000, and $3,000,000. We set the desired spending budget at 80, 90, or 100 percent of the combined after-tax income. We also assumed an asset growth rate of three percent, a dividend payment of two percent, and an interest payment of one percent. We assumed 2012 Federal and Connecticut state tax rates and brackets. Returns were homogeneous across all accounts. We solved the problem using a linear programming version of the model (i.e., a model with no logical constraints), and compared the solution to three solutions obtained using heuristics. The objective of our experiments was to compare the results of optimization against heuristics to determine the benefits of optimizing.

The experiments' formulation eliminated logical constraints, such as the calculation of the alternative minimum tax (AMT), the taxability of social security payments, and the determination to use standard or itemized deductions, by fixing a choice for each. We assumed an 85 percent taxability of social security distributions, which is the worst case for social security payment taxability. We did not consider itemized deductions, which benefit people paying high taxes, large donors, and those with large medical bills. We eliminated the AMT, which affects people paying high state and property taxes.

The first heuristic is the no-conversion strategy; all excess income goes into IRA and/or 401(k) accounts and then into taxable accounts. The second heuristic aggressively converts IRA assets into Roth IRA assets and contributes only to Roth IRA and/or Roth 401(k) accounts. At age 60, all IRA accounts are converted to Roth IRA accounts and the taxes are paid from the assets in the IRA if they are not available outside the IRA. This strategy is referred to as the full-conversion strategy. The third heuristic is the same as the second, with the exception that only outside funds are used to

convert IRA assets to Roth assets; this is the outside-funds strategy. The heuristic methods produce solutions comparable to those a customer can expect from an accountant or online calculator.

We filtered the test cases in which the optimal shortfall is greater than zero. A zero shortfall represents the cases in which meeting the spending goals given in the model input is possible. This filter left over 90,000 cases out of the original 400,000. For each test case, we compared the optimal value against the best value from the heuristics. The difference between the optimization model output and the heuristic results is because of increases in the estate's NPV. The best incumbent value was given by the no-conversion strategy 7.58 percent of the time, by the full-conversion strategy 0.13 percent of the time, and by the outside-funds strategy 92.29 percent of the time. This means that whenever outside funds were available to pay for the conversion, converting assets was almost always the best strategy, and agrees with the earlier work of Dammon ([2009]).

We define the benefit of the optimization model as the increase in the estate's NPV minus the shortfall when compared against the best heuristic value. In general, increasing initial IRA values and decreasing initial age improved the benefits of optimization over heuristics. With initial IRA values, the benefits of optimization began to plateau or even decrease as the amount in the initial IRA passed $1,000,000. This is the result of reduced tax-bracket flexibility as the amount of additional income and conversions forces taxes to go into higher and higher tax brackets. Benefits ranged from $19 (because of numeric tolerance issues where the objective values were in the millions) to $473,485. The average benefit, measured by the increase in the estate's NPV over the next-best incumbent strategy, was $88,544, and the median was $78,393. When translated into the expected increased value of the estate, the numbers are even more compelling. The average estate increased by $906,934 and the median increase was $628,114; however, the lowest increase (biggest decrease) in estate size was $13,474,425. This is because of the extra value placed on Roth assets and the excessive influence of end-of-horizon values on the results. For this case, the improvement in NPV was $296,586. The test case involves a 30-year-old.

Benefits vs. initial IRA assets and initial age
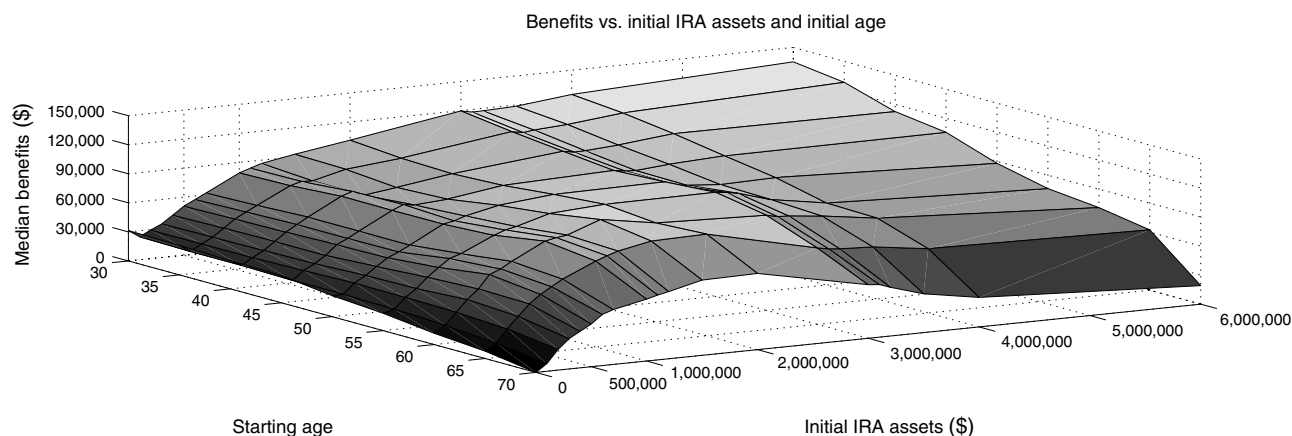
Figure 1: The improvement in estate NPV offered by optimization increases with as initial wealth and time to retirement increases. The graph shows a declining benefit at older ages as a result of the constraints caused by required minimum distributions.

The growth in assets over a 70-year period is approximately 5,807 percent, and this type of growth more than compensates for the low probabilities associated with events 70 years into the future. As the initial age in the test cases increases, the worst comparison value decreases. Figure 1 shows the average NPV improvement over heuristics plotted against initial age and initial IRA assets.

We examined cases in which optimization resulted in benefits of less than $1,500; $1,500 is an estimate for the cost to have an accountant generate a Roth conversion and retirement asset-drawdown strategy. Of over 90,000 cases, only 298 fit this criterion, and all of these began under extreme circumstances, such as having an advanced starting age (65 or 70) and (or) no IRA assets or no taxable assets. However, in 99.67 percent of the cases, an optimized plan was better than any heuristic by at least $1,500, indicating that using optimization is better than using a simple heuristic for most people. As rules on contributions, distributions, distributions from inherited accounts, and taxes change, an optimized solution that considers all the applicable rules for every year will continue to outperform the best incumbent strategy.

We analyzed the first-year results of the 90,000 test cases where the desired expenses were feasible given the initial conditions, income, and assumed returns; we show the results in Tables 3 and 4. Table 3 shows the first-year recommendations, broken out by initial

age. The percentage of IRA assets that are converted begins at a conservative 20 percent and decreases by only 10 percent as the starting age increases to 65. Intuitively, younger people have more years over which they may convert their IRAs to Roth equivalents. The amount of taxable assets sold to pay the taxes for a Roth conversion hover around 20–23 percent, and the more surprising result is that the percentage of the maximum Roth contribution tends to increase with age. This is likely to be the result of having more time to convert the assets in later years for the younger savers, while the older savers need to immediately have the assets in the Roth account. The data for age 70 are for people who are already retired.

Age	IRA converted to Roth (%)	Taxable assets sold (%)	Max. Roth contribution made (%)
30	20.30	23.11	24.50
35	19.85	22.89	25.00
40	19.01	22.32	25.63
45	17.69	21.09	27.70
50	15.64	20.78	30.65
55	13.30	20.21	35.36
60	10.74	21.22	48.20
65	9.18	19.70	44.46
70	14.64	21.36	N/A

Table 3: As age increases, the recommended average first-year conversion decreases and the amount to contribute directly to a Roth IRA increases. Required minimum distributions begin at age 70 and a new set of rules applies.

Age	IRA converted to Roth (%)	Taxable assets sold (%)	Max. Roth contribution made (%)
20.00	10.67	18.74	36.21
20.50	22.47	5.42	45.67
30.00	2.43	19.70	5.13
30.50	11.59	19.70	38.69
33.50	10.77	8.86	35.97
34.00	15.46	23.45	34.68
39.00	24.00	25.95	20.53
39.50	3.93	52.18	11.66
41.50	50.81	48.77	29.82
41.70	39.23	64.59	21.04

Table 4: The recommended percentage of an IRA to convert and percentage of taxable assets to sell generally increase, and the percentage of contributions to a Roth decreases as the combined tax bracket increases.

Their IRA percent conversion reflects the impending required minimum distributions beginning the following year, and the higher probability of mortality in the coming years.

The data in Table 4 show the average recommendations by starting combined tax bracket. Some of the results are counter-intuitive: The percent of the IRA that is converted tends to increase with the marginal rate. Part of this results from the correlation with expenditures; the larger the income, the larger the tax bracket and the larger the annual budget. The larger the budget, the larger the IRA distributions and retirement tax rate will be. Less surprising are the recommendations for selling the taxable assets. The higher the tax bracket, the lower the return on the assets in the taxable account. Finally, the percentage contribution to Roth tends to decrease with increasing tax rates, which is in line with expectations. The Roth contribution numbers are not monotonically decreasing, suggesting that other factors play a role in the decision to convert an IRA to a Roth, and in making Roth contributions.

In the remaining cases (about 310,000), the desired budget exceeds what is possible, given the income, initial assets, and initial IRA. In these cases, the benefit is measured only by the improvement in the budget shortfall. This value ranged from −$2,666 to $26,351,948; recall that shortfall values are only reduced by probability because their values represent current dollars. The 385 cases in which the benefits are negative result from constraints in the model that do not allow enough flexibility in decreasing the shortfall from one age combination to the next. The risk in relaxing those constraints is that initial budgets will be too low or might go to zero.

Originally, the optimization model always outperformed the heuristic when no constraints on shortfall were present. Typically, the results would recommend $0 consumption in initial years to build up enough savings to live at the desired lifestyle in future years. We added constraints that forced the consumption to not increase. This lead to a few situations in which the heuristic could outperform the optimization model; this occurred mainly when the spending budget was too aggressive for the initial income, but income growth allowed the heuristic model to eventually catch up. We changed the constraints again to allow decreases in shortfall by as much as the estimated after-tax increase in income from one year to the next year. That means that if after-tax income increases by $2,000 in one year, then the shortfall may decrease by as much as $2,000 in that year.

Sensitivity

The model contains two stochastic elements: mortality and investment returns. OFS models the former using a scenario tree that includes all but the rarest mortality scenarios. However, the model treats investment returns as deterministic; although commonplace in personal financial planning, this assumption is clearly a simplification. The inputs for portfolio returns are the percentage of the portfolio paid back as dividends, the percentage of the portfolio paid back as interest, and the growth in the value of the portfolio. To test the viability of using a deterministic set of returns for these parameters, we created a set of values for each of them, which we derived from S&P Case-Shiller data (Shiller 2014). We calculated the average return, dividend yield, and long-term interest rate over each 40-year span in the data, starting from 1,871 and ending in the present. We took the highest and lowest values and created an evenly spaced range of 11 values; we then divided the values by two to reflect a 50–50 allocation between stocks and bonds.

The parameter values used for the growth rate ranged from −0.19 percent to 4.28 percent, the values for the interest rate ranged from 1.50 percent to 3.72 percent, and the parameter values for the dividend rates ranged from 1.49 percent to 2.87 percent.
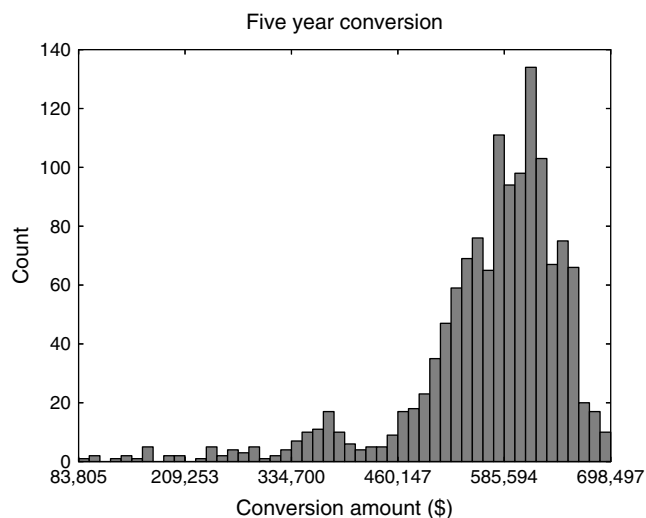
Figure 2: Five-year conversion amounts in our sensitivity example indicate that the model may suggest slightly overconverting assets. Current tax rules allow for a recharacterization (reversing the conversion); thus, overconversion is not harmful to the client.

We used sample customer parameters of a 45-year-old couple, each earning $160,000 annually, and each with $200,000 in his or her IRA, nothing in a Roth account, and $100,000 in a taxable account.

We tested the sensitivity of the first five years of conversions; the five-year cumulative conversion amount smooths variations in conversions that can occur in the first few years. Figure 2 shows a histogram of the model recommendations. The average five-year conversion over all value combinations is $567,533, with a standard deviation of $94,083 and a median value of $588,470, indicating that the conversion amount is roughly consistent across many parameter combinations. Furthermore, most outlying values are above the mean, indicating that when the model's recommendation is not optimal for the actual parameter values, the recommended conversion will be higher. In particular, the five-year conversion amount using the average parameter values (the mid-points of each range) is $659,488, and only 8.6 percent of the scenarios convert more. This is not a major concern because the IRS allows recharacterizations of Roth conversions (i.e., a taxpayer has the ability to undo a conversion and revert the funds); some retirement tax experts actually recommend overconverting and then recharacterizing (Keebler and Bigge

2007). A recommended conversion amount significantly lower than the optimum would be more problematic for the customer, because it would imply a lost savings opportunity.

Case Study

Using a purely continuous model (with fixed logical decisions) over a mixed-integer formulation has several drawbacks. With the introduction of binary indicator variables, it is possible to include constraints to determine the exact taxability of social security payments, the AMT, the decision to use the itemized or the standard deduction, and of most recent importance, the calculation of the healthcare surtax. This surtax is a 3.8 percent tax on all investment income when income exceeds $250,000 (when married and filing jointly). Our results show acceleration in Roth conversions to reduce taxable assets, which will be subject to the tax for those whose income exceeds the threshold, with conversions stopping when income reaches this threshold.

In practice, we often run several scenarios to understand the model's sensitivity to various parameters. One client case involved a California retiree and his wife with approximately $2,800,000 in an IRA and $3,000,000 in taxable assets. The client was considering converting the entire IRA and wanted to know if this was the best strategy. We created two scenarios for the client, one with existing tax rates and the healthcare surtax, and a second with the healthcare surtax and higher Clinton-era tax rates. In the lower tax-rate scenario, the model suggested converting approximately $1,600,000 of the IRA and converting the remainder over the next 14 years. The model with the higher tax rates accelerated the 2012 conversion to $2,472,000. The benefits of optimization were between $173,480 in the higher-rate scenario and $175,800 in the base scenario, measured in increased estate NPV. The best incumbent heuristic in both scenarios was the full conversion.

In the 2012 conversion, the client's marginal tax rates were at the 35 percent rate and he was at the top of the California tax rate. The taxes for converting the additional $400,000 of the IRA were about $222,000. The effective federal tax rate on the remaining conversion was between 26.51 percent and 31.56 percent,

lower than the 2012 marginal rate. By not effecting a complete conversion, the client is able to keep an additional $222,000 in assets and will pay a lower tax rate over the next six years on the remaining conversions. In both scenarios, the IRA is eventually completely converted because the client does not need the money from the IRA, and using up the inefficient taxable assets first is a better option. Ultimately, the client will still have as much in the Roth IRA as if the conversion had been done only in the first year, but the present value of the assets left in the taxable account after taxes will be approximately $173,000 higher.

We had to estimate what the tax rates would be under a higher Clinton-era scenario. However, this conversion represents what we might view as an upper bound on conversion amounts, while the conversion amount when using the existing rates is a lower bound. The client ultimately decided to convert using the upper-bound amounts and to take advantage of recharacterization in 2013 to adjust the optimal conversion to the reality of what tax rates will be in 2013 and beyond. In recharacterization, a converted IRA may be switched back from a Roth to a regular IRA. It may be a partial recharacterization, and we also used optimization here. The final 2012 conversion was approximately $650,000. It was a level with which the client felt comfortable, and the estate's NPV was within two percent of the optimal.

Another benefit of using the model is that it indicates the client will be paying the AMT in the next three years under the higher-rate scenario. The AMT does not allow for deductions in taxes paid. The model projected 2013 California state taxes to be $14,770. If the client prepaid those taxes in 2012, he would be able to deduct them in 2012, when his marginal rate was already at 35 percent. This represents an extra $5,170 in tax savings. The risks are that the taxes could be lower, that his federal marginal rate in 2013 could still be 35 percent, and that he would still be able to claim a deduction in 2013 for the state taxes paid in that year.

We wanted to know what the effect of all the potential taxes would be on our client. We examined a scenario using the existing tax rates, removing the healthcare surtax, and treating all dividends as qualified instead of as ordinary income. We compared the NPV objective values for both cases. From the base case to the higher-rate scenario, the NPV value decreased by $48,312. If the higher rates come into effect, for example because of the fiscal-cliff crisis, this represents a 1.36 percent tax on the wealth our client will pass to his heirs.

Conclusions

By modeling Roth conversions, retirement contributions, retirement distributions, and spending using stochastic linear programming and mixed-integer programming techniques, we can outperform the traditional manual techniques used for evaluating Roth conversions and distributions. With linear programming, we can simultaneously evaluate the effects of a conversion of any size in any year. Although the possibilities are infinite, mathematical programming solvers can easily handle them. The best alternative is manually plugging in the conversion amounts over time into a financial planning calculator. This is a trial-and-error method, and no published techniques are available to recommend amounts in different years. The result is an increase in wealth; broadly speaking, our experiments indicate this benefit decreases with age and improves with increased initial IRA assets.

Stochastic linear programming techniques allow us to evaluate mortality possibilities. This creates a plan that considers the possibility of the death of a spouse, and makes decisions that protect the surviving spouse. The results corroborate the theory that conversions should be done at an early age when marginal tax rates are low. In one test case, however, the IRA owner had a low income, high savings rate, was only 30 years old, and had no retirement savings. Rather than immediately making contributions to a Roth IRA, the model suggested waiting more than 10 years before making any conversions. The reason was that the amount of money left over for the surviving spouse, who had no other income, would have been lower than if the working spouse had made Roth contributions or conversions. With the low levels of savings, paying a small amount of taxes on the distributions rather than a larger amount on the contributions was a better strategy. This suggests that the client might benefit from insurance products, so that he could risk conversions and still provide for a surviving spouse.

One final conclusion is that although optimization suggests a strategy, we always need to evaluate multiple scenarios. This determines the sensitivity to different assumptions of future tax rates and investment returns. Another reason for multiple scenarios is to familiarize the client with the options available and ensure that the client is comfortable with the decisions. We often change the variables or bounds on the constraints to force decisions, such as the recharacterization percentage or the Roth distributions, to understand the effects of those decisions on shortfall and NPV. In some cases, clients may have their own levels of comfort, such as the amounts they would like to convert. We can test the effects of this forced decision and compare the resulting objective value to the (unconstrained) value. Some decisions may only affect an estate's value by one percent; however, others, such as never taking Roth distributions, may drastically affect both estate value and shortfall. Although optimization can significantly increase both spending and wealth, the client must be comfortable with the strategy and be able to implement it.

Acknowledgments

Appendix. Basic Model Formulation

The following formulation details a basic version of the optimization model used by OFS. The notation follows the recommendations in Teter et al. (2015).

Indices and Sets

$t \in T$: the set of spouse age combinations t.

$x \in \{1, 2, 3\}$: the type of tax types x. 1 is for state, 2 is for federal, 3 is for federal capital gains.

$r \in R_{t,x}$: the set of tax rates r imposed during a given age combination t for a given type of income x.

$p \in \{1, 2\}$: the indicator in an age combination t is the index refers to data for spouse 1 or 2.

$d \in T$: the set of all age combinations where neither spouse has outlived the other (the diagonal).

$h \in \{0, 1\}$: the indicator of whether an account is not inherited (0) from a spouse or is (1).

$e \in \{0, 1\}$: the employer-sponsored indicator of whether an account is individual (0) or employer sponsored (1).

$i \in \{1, 2, 3\}$: the indicator of whether an account is taxable (1), tax deferred (2), or tax free (3).

Parameters

P_t^m: the probability the couple dies at age combination t.

P_t^s: the probability at least one spouse survives to age combination t.

C_t: the discount factor used to discount values in age combination t to the present.

W_i: the adjustment to the value of inherited accounts used to reflect the treatment of distributions as income.

M_t: the expected earned income in age combination t.

A: the taxable-basis percentage of the taxable assets.

D: the dividend rate paid on investments.

I: the interest rate paid on investments.

G: the growth rate of investments.

E_t: the after-tax budget for age combination t.

O_t: the social security payment made during age combination t.

$B_{t,x,r}$: the tax bracket size for age combination t for tax type x for rate r.

$L_{t,h}$: the life expectancy used in age combination t, which is also dependent on the inheritance status h.

N: the percentage of the IRA that was made with nondeductible contributions.

$Y_{t,e,i}$: the limit on contributions in age combination t for account type i for a personal or employer-sponsored e account.

U_t: the maximum combined deduction and exemption allowed in age combination t of type n.

V_t: the expected increase in after-tax income based on expected earnings in age combination t.

R: the planned retirement age.

Q: the maximum earned income allowable before Roth contributions are disallowed.

F: the probability adjusted worse-case shortfall, which is the result of the first pass through the optimization.

Decision Variables

s_t: the difference between the budget and actual spending in age combination t.

$a_{t,h,e,i,p}$: the value of account in age combination t with inheritance status h, sponsor indicator e, account type i, and, in the case in which both spouses are alive (the diagonal), the owner p.

c_t: the taxable income in age combination t.

$s_{t,h,e,i,p}$: the distribution or sales of assets in age combination t with inheritance status h, sponsor indicator e, account type i, and, in the case in which both spouses are alive, the owner p.

$u_{t,h,e,i,p}$: the contribution or purchase of assets in age combination t with inheritance status h, sponsor indicator e, account type i, and, in the case in which both spouses are alive, the owner p.

$v_{t,e,p}$: the conversion from tax-deferred to tax-free accounts in age combination t from and to employer-sponsored account e for owner p.

f_t: the expected shortfall in age combination t.

$b_{t,x}$: the taxes in age combination t for tax type x.

$m_{t,x,r}$: the income in age combination t for tax type x in bracket r.

n_t: the deduction used in age combination t.

Objective Functions

The recommendations stem from optimizing two objectives in succession. The first objective minimizes the budget shortfall, given by

$$\min \sum_{t \in T} f_t \cdot P_t^s \quad (= F). \tag{1}$$

Next, we change the objective to maximize the net present value and constraint the shortfall to the value F obtained in the first pass through the optimization:

$$\max \sum_{t \in T} \sum_{h \in \{0,1\}} \sum_{e \in \{0,1\}} \sum_{i \in \{1,2,3\}} \sum_{p \in \{1,2\}} P_t^m \cdot C_t \cdot W_i \cdot a_{t,h,e,i,p} \tag{2}$$

$$\text{s.t.} \sum_{t \in T} f_t \cdot P_t^s \leq F \tag{3}$$

Constraints

$$c_t = M_t + \sum_{h \in \{0,1\}} \sum_{e \in \{0,1\}} \sum_{i \in \{2\}} \sum_{p \in \{1,2\}} s_{t,h,e,i,p}$$

$$+ \sum_{h \in \{0,1\}} \sum_{e \in \{0,1\}} \sum_{i \in \{1\}} (1-B) \cdot s_{t,h,e,i}$$

$$+ \sum_{h \in \{0,1\}} \sum_{e \in \{0,1\}} \sum_{i \in \{1\}} (D+I) \cdot a_{t-1,h,e,i}$$

$$- \sum_{e \in \{0,1\}} \sum_{p \in \{1,2\}} u_{t,0,e,2,p}, \quad \forall t \in T; \tag{4}$$

$$\sum_{e \in \{0,1\}} \sum_{i \in \{1,2,3\}} \sum_{p \in \{1,2\}} u_{t,0,e,i,p} + E_{ac} - f_t \sum_{x \in \{1,2,3\}} b_{t,x}$$

$$= \sum_{h \in \{0,1\}} \sum_{e \in \{0,1\}} \sum_{i \in \{1,2,3\}} \sum_{p \in \{1,2\}} s_{t,h,e,i,p} + M_t + O_t$$

$$+ \sum_{h \in \{0,1\}} \sum_{e \in \{0,1\}} \sum_{i \in \{1,2,3\}} \sum_{p \in \{1,2\}} s_{t,h,e,i,p}$$

$$+ \sum_{h \in \{0,1\}} \sum_{e \in \{0,1\}} \sum_{i \in \{1\}} (D+I) \cdot a_{t-1,h,e,i}, \quad \forall t \in T; \tag{5}$$

$$b_{t,x=3} = (D \cdot a_{t-1,0,0,0} + (1-B) \cdot s_{t,0,0,0}) \cdot R_{t,x=3}, \quad \forall t \in T; \tag{6}$$

$$\sum_{r \in R_{t,x}} m_{t,x,r} = c_t, \quad \forall t \in T,\ x \in \{1\}; \tag{7}$$

$$\sum_{r \in R_{t,x}} m_{t,x,r} = c_t - D \cdot a_{t-1,0,0,1} - (1-B) \cdot s_{t,0,0,1}$$

$$- n_t + 0.85 \cdot O_t, \quad \forall t \in T; \tag{8}$$

$$\sum_{r \in R_{t,x}} r \cdot m_{t,x,r} = b_{t,x}, \quad \forall t \in T,\ x \in \{1,2\}; \tag{9}$$

$$s_{t,0,e,2} \geq \frac{a_{t-1,0,e,2}}{L_{t,0}}, \quad \forall t \geq 70,\ \forall e \in \{0,1\}; \tag{10}$$

$$s_{t,1,0,i} \geq \frac{a_{t-1,1,0,i}}{L_{t,1}}, \quad \forall t \in T,\ \forall i \in \{2,3\}; \tag{11}$$

$$a_{t,0,0,1} = (1+G) \cdot a_{t-1,0,0,1} + u_{t,0,0,1} - s_{t,0,0,1}, \quad \forall t \in T; \tag{12}$$

$$s_{t,0,0,1} \leq (1+G) \cdot a_{t-1,0,0,1}, \quad \forall t \in T; \tag{13}$$

$$a_{t,h,e,i} = (1+G+D+I) \cdot a_{t,h,e,i} - s_{t,h,e,i} + u_{t,h,e,i},$$
$$\forall t \in T,\ h \in \{0,1\},\ e \in \{0,1\},\ i \in \{2,3\}; \tag{14}$$

$$a = b + c, \quad \forall t \in T; \tag{15}$$

$$a_{t,0,e,i=3,p} \geq (1+G+D+I) \cdot a_{t-1,0,e,i=3,p}$$
$$+ u_{t,0,e,i=3,p} - s_{t,0,e,i=3,p} + v_{t,e,p},$$
$$\forall t \in T, \text{ s.t. } (t-1) \in d,\ e \in \{0,1\},$$
$$p \text{ is surviving spouse}; \tag{16}$$

$$\sum_{h \in \{0,1\}} a_{t,h,e,i=3,p} = (1+G+D+I) \cdot \sum_{h \in \{0,1\}} a_{t-1,h,e,i=3}$$
$$+ u_{t,0,e,i=3,p} + v_{t,e,p} - \sum_{h \in \{0,1\}} s_{t,h,e,i=3,p},$$
$$\forall t \in T, \text{ s.t. } (t-1) \in d,\ e \in \{0,1\},$$
$$p \text{ is surviving spouse}; \tag{17}$$

$$u_{t,0,0,i=3} = 0, \quad \forall t \geq 70; \tag{18}$$

$$\sum_{i \in \{2,3\}} u_{t,0,e,i} \leq \min\{M_t, Y_{t,e,i}\}, \quad \forall t \leq R,\ e \in \{0,1\}; \tag{19}$$

$$u_{t,0,0,2} = 0, \quad \forall t \in T, \text{ s.t. } M_t \geq Q; \tag{20}$$

$$f_t \geq f_{t-1} - V_t, \quad \forall t \in T,\ t > 1; \tag{21}$$

$$a_{t,h,e,i,p},\ s_{t,h,e,i,p},\ u_{t,h,e,i,p} \geq 0,$$
$$\forall t \in T,\ \forall h \in \{0,1\},\ \forall e \in \{0,1\},$$
$$\forall i \in \{1,2,3\},\ \forall h \in \{1,2\};$$

$$c_t \geq 0, \quad \forall t \in T;$$

$$b_{t,x} \geq 0, \quad \forall t \in T,\ \forall x \in \{1,2,3\};$$

$$v_{t,e,p} \geq 0, \quad \forall t \in T,\ \forall e \in \{0,1\},\ \forall p \in \{1,2\};$$

$$0 \leq m_{t,x,r} \leq B_{t,x,r}, \quad \forall t \in T,\ \forall x \in \{1,2,3\},\ \forall r \in R_{t,x};$$

$$0 \leq f_t \leq E_t, \quad \forall t \in T;$$

$$0 \leq n_t \leq U_t, \quad \forall t \in T.$$

References

Ashby DJ, Williams DG, Stinson TA (2010) Investment strategies for tax diversification: Don't put all your eggs in one basket. *J. Bus. Econom. Res.* 8(1):115–122.

Bergstresser D, Poterba J (2004) Asset allocation and asset location: Household evidence from the survey of consumer finances. *J. Public Econom.* 88(9–10):1893–1915.

Bernheim BD, Forni L, Gokhale J, Kotlikof LJ (2000) How much should Americans be saving for retirement? *Amer. Econom. Rev.* 90(2):288–292.

Burman LE, Gale WG, Weiner D (2001) The taxation of retirement saving: Choosing between front-loaded and back-loaded options. Accessed January 30, 2014, http://dx.doi.org/10.2139/ssrn.272145.

Dammon RM (2009) Retirement investing: Analyzing the Roth conversion option. Technical Report 104, Carnegie Mellon University, Tepper School of Business, Pittsburgh.

Dammon R, Poterba J, Spatt C, Zhang H (2005) Maximizing long-term wealth accumulation: It's not just about "what" investments to make, but also "where" to make them. Technical Report 108, Carnegie Mellon University, Tepper School of Business, Pittsburgh.

Davidoff T, Brown JR, Diamond PA (2003) Annuities and individual welfare. Technical report, National Bureau of Economic Research, Cambridge, MA.

Gokhale J, Kotlikoff L, Neumann T (2001) Does participating in a 401(k) raise your lifetime taxes? Working paper, National Bureau of Economic Research, Cambridge, MA.

Hoyt CR (2010) Rethinking Roth IRA conversions in 2010. *Probate & Property* (September–October):12–16.

Hungerford TL, Gravelle JG (2010) Individual retirement accounts (IRAs): Issues and proposed expansion. Technical report, Congressional Research Service, Washington, DC.

Internal Revenue Service (2014) Publication 590-A. Contributions to Individiual Retirement Arrangements (IRAs). Department of Treasury, Internal Revenue Service, Washington, DC. Accessed February 22, 2016, https://www.irs.gov/pub/irs-pdf/p590a.pdf.

Kaplan RL (2009) To Roth or not to Roth: Analyzing the conversion opportunity for 2010 and beyond. *Daily Tax Rep.* 9(181):1–4.

Keebler R, Bigge S (2007) To convert or not convert, that is the question. *J. Retirement Planning* 10(1):37–46.

Milevsky M, Moore K, Young V (2006) Asset allocation and annuity-purchase strategies to minimize the probability of financial ruin. *Math. Finance* 16(4):647–671.

Poterba JM, Shoven JB, Sialm C (2000) Asset location for retirement savers. Working paper, National Bureau of Economic Research, Cambridge, MA.

Poterba J, Venti S, Wise D (2011a) The composition and drawdown of wealth in retirement. *J. Econom. Perspect.* 25(4):95–117.

Poterba JM, Venti SF, Wise DA (2011b) The drawdown of personal retirement assets. Working paper, National Bureau of Economic Research, Cambridge, MA.

Rockafellar RT, Uryasev S (2000) Optimization of conditional value-at-risk. *J. Risk* 2(3):21–41.

Rockafellar RT, Uryasev S (2002) Conditional value-at-risk for general loss distributions. *J. Banking Finance* 26(7):1443–1471.

Sen S, Higle JL (1999) An introductory tutorial on stochastic linear programming models. *Interfaces* 29(2):33–61.

Shiller R (2014) Online data—Robert Shiller. Accessed September 30, 2014, http://www.econ.yale.edu/~shiller/data.htm.

Shoven JB, Sialm C (2004) Asset location in tax-deferred and conventional savings accounts. *J. Public Econom.* 88(1):23–38.

Slott E (2012) *The Retirement Savings Time Bomb . . . and How to Defuse It* (Penguin Books, New York).

Teter MD, Newman A, Weiss M (2015) Consistent notation for presenting complex optimization models in technical writing. *Surveys Oper. Res. Management Sci.* Forthcoming.

U.S. Social Security Administration (2007) Actuarial life table. Accessed February 9, 2016, http://www.socialsecurity.gov/OACT/STATS/table4c6.html.

VanZante NR, Wagman GR, Fritzsch RB (2005) Tips for maximizing the value of your retirement investment portfolio. *J. Bus. Econom. Res.* 3(12):35–40.

Yang X (2010) Applying stochastic programming models in financial risk management. Doctoral dissertation, University of Edinburgh, Edinburgh, UK.

Verification Letter

Marshall A. Glick, President, Law Offices of Marshall A. Glick, 6345 Balboa Boulevard, Suite 1-300, Encino, California 91316-1523, writes:

"I am writing to describe my experience with Optimized Financial Systems of Canton, MI. I am the President of Marshall A. Glick, a professional corporation. In my law practice I emphasize estate planning, non-profit law, business and corporate law, and real estate law. I worked with Optimized Financial Systems in 2012 to prepare for the consequences of the 'fiscal cliff' occurring after December 31, 2012, and in connection with converting my several million dollar traditional IRA to a Roth IRA. I had initially been advised to do a full Roth IRA conversion and, on the recommendation of a colleague, I sought a second opinion from Optimized Financial Systems. Based upon the recommendation from Optimized Financial Systems, I initially converted about 75% of my traditional IRA to a Roth IRA.

"In early 2013 I continued to work with Optimized Financial Systems to recharacterize some of my 2012 Roth IRA conversion back to a traditional IRA. Based upon updated asset amounts, market movements, the absence of the fiscal cliff, a revised budget, and the advice I received from Optimized Financial Systems, I decided to recharacterize 75% of the 2012 conversion back to a traditional IRA.

"I plan on leaving the remainder of the Roth IRA untouched as part of my estate I plan on leaving to my children. Optimized Financial Systems explained the consequences of failing to make further conversions and not taking distributions from my Roth IRA. As a result, I have updated my IRA conversion plans to maximize the long term value of my estate. I feel very fortunate to have found Optimized Financial Systems and they have assisted me greatly in my long term family estate planning, giving me the kind of detailed and ongoing advice that I had been unable to obtain elsewhere. Optimized Financial Systems was always immediately responsive to my questions and of great assistance in enabling me to make what I believe to be the best possible decisions regarding the conversion of my traditional IRA accounts to a Roth Account. In my judgment they are truly exceptional."

Joshua Woodruff serves as Optimized Financial System's President and Modeling Director. He previously held positions at IBM, Morgan Stanley, and ILOG focusing on modeling and optimization in various financial applications. Josh is pursuing a PhD at the University of Texas at Austin in the Operations Research department, has a BA in economics and a BS in applied math from Yale University, and an MS in industrial engineering from the University of Southern California.

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Alejandro Toriello is an assistant professor in industrial and systems engineering at Georgia Tech, where he also obtained BS and PhD degrees in industrial engineering in 2003 and 2010, respectively. He was previously on the Industrial and Systems Engineering faculty at the University of Southern California. His research interests include discrete and dynamic optimization and their applications.