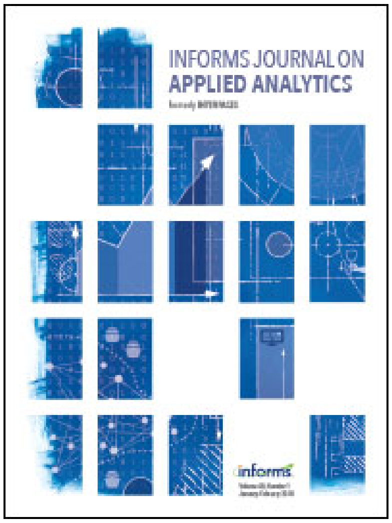

INFORMS Journal on Applied Analytics

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

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INFORMS JOURNAL ON APPLIED ANALYTICS
Articles in Advance, pp. 1–17
ISSN 0092-2102 (print), ISSN 1526-551X (online)
http://pubsonline.informs.org/journal/inte

Improving Sports Media's Crystal Ball for National Basketball Association Playoff Elimination

Mark A. Husted,[a] **Eli V. Olinick,**[b] **Alexandra M. Newman**[a]

[a] Department of Mechanical Engineering, Colorado School of Mines, Golden, Colorado 80401; [b] Department of Engineering Management, Information, and Systems, Bobby B. Lyle School of Engineering, Southern Methodist University, Dallas, Texas 75205

Contact: mhusted@mines.edu (MAH); olinick@lyle.smu.edu, https://orcid.org/0000-0001-7856-1826 (EVO); anewman@mines.edu, https://orcid.org/0000-0001-9886-9721 (AMN)

Received: May 1, 2019
Revised: January 1, 2020
Accepted: January 2, 2020
Published Online in Articles in Advance: July 20, 2020

https://doi.org/10.1287/inte.2020.1034

Abstract. The National Basketball Association (NBA) is divided into two conferences, each of which comprises 15 teams. At the end of the regular season, the top eight teams from each conference, based on winning percentage, compete in the playoffs. Mixed-integer-programming (MIP) models determine when a team has guaranteed its position in the playoffs (clinched) or, conversely, when it has been eliminated before the completion of the regular season. Our models incorporate a series of complex two-way tiebreaking criteria used by the NBA to determine how many more games are needed either to clinch or to avoid elimination. We compare the time at which a given team has clinched or been eliminated, in terms of the number of games played in the season to date, as posted in the NBA official standings, against results from our mixed-integer program. For the 2017–2018 season, when our models outperform those of the NBA, they do so by an average of 4.1 games. We also describe a scenario in which the NBA erroneously reported that the Boston Celtics had clinched a playoff spot and, conversely, show that the Golden State Warriors had clinched a playoff spot before the official announcement by the NBA.

History: This paper was refereed.

Keywords: integer programming applications • basketball • tiebreaking • playoffs

Introduction

Standings in the major professional athletics leagues are published daily in newspapers and updated in real time on sports websites. These outlets typically report when teams are *eliminated* from the playoffs (i.e., have fallen so far behind that they cannot qualify even if they win all of their remaining games) and so-called "magic numbers" that represent how close the leading teams are to the opposite condition, that of *clinching* a playoff spot. As noted by Adler et al. (2002, p. 12), "Fans of professional sports teams have an insatiable desire for information about the performance of their favorite teams." Fan interest in the standings helps drive traffic to commercial websites and determines which games to televise nationally. Teams have additional interest in this information because they can start selling tickets for potential home-court playoff games as soon as they have clinched a spot (NBA.com 2018b). Accurate assessment of teams' playoff chances can also help coaches plan playing time for injured or rookie athletes. Sports elimination problems are popular examples in math programming pedagogy (e.g., Schrage 1984 and Ahuja et al. 1993), because many students find them inherently interesting and easy to understand (Robinson 1991).

The Basketball Association of America was founded in 1946 and merged with the National Basketball League in 1949 to form the National Basketball Association (NBA). The number of teams in the association fluctuated until the 1970s when the NBA started to grow more consistently. In 2004, the league expanded to the 30 teams present today, located across the Unites States of America and Canada. These teams are divided into two different conferences, East and West. Every year, each team plays 82 games in the regular season, at the end of which the top eight teams from each conference qualify for a postseason tournament, or the playoffs, which, at the time of this writing, is a seven-game series matchup in which the losing team is eliminated and the winning team continues to another seven-game playoff series (through the finals). The winning team of the playoffs is declared the league champion for that year and awarded the Larry O'Brien Championship Trophy.

Literature Review

Until relatively recently, sports media have primarily relied on a simple calculation to determine whether a team has been eliminated from first place—that is, whether it has fewer games left to play than the number of games it would need to win to match the record of the current first-place team. Sports media have traditionally reported a statistic, known as the magic number, for teams that have not been eliminated by this simple calculation. The traditional magic number is the smallest number such that any combination of wins by the first-place team and losses by the second-place team totaling the magic number guarantees that the first-place team finishes the season with a better record than the second-place team (Adler et al. 2002). Applying the maximum-flow/minimum-cut theorem, Schwartz (1966) shows that although this is a sufficient condition to determine elimination, it is not necessary. Hoffman and Rivlin (1971) show that the problem of determining elimination from first place, given the current league standings and schedule of remaining games, can be solved by a single maximum-flow calculation. Robinson (1991) shows that an equivalent linear programming approach would have determined elimination in the 1987 Major League Baseball (MLB) season an average of three days—and as many as five days—earlier than the simple, traditional calculation.

By introducing additional constraints, Adler et al. (2002) extend the maximum-flow formulation of Schwartz (1966) to derive integer-linear programming models to determine the minimum number of games a given team must win to clinch first place, or a playoff spot, and also to avoid elimination from the MLB playoffs. These numbers have been calculated and posted on the RIOT website (https://s2.smu.edu/~olinick/riot/baseball_main.html) since 1996 (Adler et al. 1996). Lucena et al. (2008) describe an agent-based software system used to implement a similar website called FutMax that published elimination and clinch numbers calculated by solving models developed by Ribeiro and Urrutia (2005) for the Brazilian national football (soccer) championship. During the 2002 season, Ribeiro and Urrutia (2005) used this methodology to show that sports media were prematurely claiming that certain teams had clinched playoff spots. Cheng and Steffy (2008) present integer-linear programming models to determine clinching and elimination numbers for teams in the National Hockey League (NHL). Applying their models to the 2003–2004 season, they were able to detect a playoff clinch two days earlier and an elimination one day earlier than the simple rules used by the media.

Each of these leagues poses unique modeling challenges. For example, the NHL and Brazilian soccer league use point systems to determine how teams are ranked in the standings. NHL teams are awarded two standings points for a win and one point for a loss in overtime (NHL.com 2019), whereas Brazilian soccer teams are awarded three standings points for a win and one point for a tie (Wikipedia 2019). Ties for MLB playoff berths are resolved by additional games between the tied teams, whereas the NBA uses the statistical tiebreaking criteria given in the description of our mixed-integer programs. Russell and van Beek (2008; 2009a, b) apply constraint programming techniques to address the NHL's tiebreaking rules, which have proven to be extremely challenging for state-of-the-art commercial MIP solvers (Cheng and Steffy 2008).

For leagues with playoff structures such as MLB, Adler et al. (2002) show that there is a minimum win threshold for each division that all teams in the division must reach to avoid elimination from first place. In addition to simplifying the calculations for the RIOT website, this result led to renewed interest in the theory of sports elimination problems. Schlotter and Cechlárová (2018) provide a recent survey of the literature related to the computational complexity of sports elimination problems. In particular, Gusfield and Martel (2002) show that determining elimination from the playoffs in multidivision leagues with "wild card" teams is $\mathcal{NP}$-complete—hence, the justification of the use of mixed integer and constraint programming in the above-mentioned work.

Websites such as https://www.playoffmagic.com/nba/conference/ and https://www.nba.com/standings provide insights into whether an NBA team has a chance to make the playoffs based on its regular season records and extend the traditional calculation. However, the fact that these sites are still prone to both false-negative and false-positive results indicates that their calculations are based on heuristics. Our contribution is a set of rigorous mathematical optimization models (see Appendices A and D) that (i) determine optimization-based magic numbers for each team in the NBA; (ii) include the mathematical logic associated with the tiebreaking rules; and (iii) solve instances efficiently in that we use linearizations, where applicable (see Appendix B), and tailor big M coefficients on logical constraints (see Appendix C). To our knowledge, these are the first such models for the NBA playoff race.

Optimization Models

There are four optimization-based magic numbers that determine a team's standings in the conference. Because it is the number from which others can easily be derived, we focus on the *playoff elimination number*—that is, the minimum number of games team k needs to win to still be in contention for the playoffs at any given date in the season. We present a corresponding mixed-integer program, $(\mathcal{M}_k)$, for each team k, for which the model must be run. The primary inputs for such a model instance are current win-loss records of the NBA teams and the schedules of games remaining in the season. Specifically, parameter $\hat{w}_i$ represents the number of games that team i has won so far, and parameter g_{ij} is the number of games left to play between teams i and j. The fundamental decision variables in the model are scenario-specific; Y_{ij} represents the number of times team i beats team j in the scenario, and $W_i = \hat{w}_i + \sum_j Y_{ij}$ is the total number of wins team i has at the end of the season. The model has a set of constraints to ensure that the scenario is feasible with respect to the schedule of remaining games (i.e., $Y_{ij} + Y_{ji} = g_{ij}$) and constraints that restrict the mixed-integer program's feasible region to scenarios in which team k is one of the eight playoff teams in its conference. The objective minimizes W_k for each k. Thus, an optimal solution to $(\mathcal{M}_k)$ determines a scenario in which team k makes the playoffs with as few wins as possible. If $(\mathcal{M}_k)$ is infeasible, then team k is eliminated from the playoffs. Otherwise, team k's playoff elimination number is $W_k - \hat{w}_k$.

The RIOT website hosts this and additional optimization models, and it correspondingly provides three other numbers for each team: (i) the first-place elimination number, which is the minimum number of games a team must win to avoid being eliminated from first place in its conference; (ii) the playoff clinch number, which represents the minimum number of games a team must win to ensure that it finishes the season among the top eight teams in the conference; and (iii) the first-place clinch number, which represents the minimum number of games a team must win to ensure that it finishes the season in first place.

Specifically, we note the following:

- The model for the first-place elimination number is similar to $(\mathcal{M}_k)$ except that the constraints restrict the feasible region to scenarios in which team k finishes in first place in its conference.
- The model for the playoff clinch number is essentially the opposite of $(\mathcal{M}_k)$. That is, it maximizes the number of games that team k wins subject to team k finishing out of contention for the playoffs. If the mixed-integer program is infeasible, then team k has clinched a playoff berth. Otherwise, team k's playoff clinch number is $(W_k - \hat{w}_k) + 1$.
- Likewise, the first-place clinch number is determined by solving a model that maximizes W_k subject to team k not finishing the season in first place.

In this paper, we refer to these four models for the NBA collectively as *RIOT*. Although we host all four optimization models on the website and use three here in our analysis, owing to space considerations, we explain in mathematical detail only the one that determines the playoff elimination number. The mixed-integer programs for the NBA are similar in nature to those solved for the MLB on the RIOT website. However, the different methods used by MLB and the NBA to resolve ties for playoff berths lead to significant differences in the technical details of the models. Whereas ties in the MLB playoff races are resolved by extending the season with additional games between the tied teams (MLB.com 2014), the NBA applies the complex hierarchy of tiebreaking criteria described in the following sections.

Two-Way Tiebreaking Criteria

Prior to 2016, the NBA prioritized divisional leaders, which were given a higher seed in the playoffs and would be likely to beat teams with which they were tied, even if the other team had a better head-to-head win percentage. However, in 2016, the NBA accepted the tiebreaking criteria in Table 1. Model $(\mathcal{M}_k)$ mathematically describes the first six criteria; criterion 7 is calculated, but the number of points a team can score in a game is theoretically unlimited. So although the criterion can be used as a last-resort tie-breaker at the end of the season, its looseness renders it generally irrelevant by midseason.

Note that these tiebreaking criteria violate necessary conditions for the existence of the first-place elimination threshold derived by Adler et al. (2002) for MLB. For example, let us assume that the team currently in first place has won 50 games and has a first-place elimination number of 0 (i.e., there exists a scenario in which the current first-place team fails to win another game but nevertheless finishes in first place). Let us further assume that the second-place team has already lost the head-to-head series with the first-place team. Because of the first tiebreaking criterion, the team that is currently in second place would have to win at least 51 games to finish in first place. Thus, it is not necessarily true that every team in an NBA conference has the same threshold for first-place elimination.

Three-or-More-Way Tiebreaking Criteria

The tiebreaking criteria between three or more teams largely mimic the rules for two-way ties with the exception

Table 1. The Tiebreaking Rules for the NBA for a Two-Way Tie (ESPN.com 2018), Ordered by Decreasing Importance, and in Which a Tie Is Broken by the First Rule in the Sequence for Which There Is a Discrepancy in Outcome Between the Two Teams

Criterion	Description of two-way tie-breaker
1	Team with the best win percentage among both tied teams
2	Division leader wins over the team not leading the division (each conference has three divisions)
3	Team with the higher division win percentage (if both tied teams are in the same division)
4	Team with the higher conference win percentage
5	Team with the higher win percentage versus the conference playoff teams
6	Team with the higher win percentage versus the other conference playoff teams (including teams tied for a playoff position)
7	Team with the higher point differential during the regular season

Table 2. The Tiebreaking Rules for the NBA for a Three-or-More-Way Tie (ESPN.com 2018), Ordered by Decreasing Importance, in Which a Tie Is Broken by the First Rule in the Sequence for Which There Is a Discrepancy in Outcome Between the Three or More Teams in Question

Criterion	Tie between three or more teams
1	Division leader wins over the team not leading the division (each conference has three divisions)
2	Team with the best win percentage among all tied teams
3	Team with the higher division win percentage (if all tied teams are in the same division)
4	Team with the higher conference win percentage
5	Team with the higher win percentage versus the conference playoff teams
6	Team with the higher point differential during the regular season

that the first two criteria are flipped in order of importance and the sixth criterion is eliminated (see Table 2).

It falls outside of the scope of our study to incorporate any three-or-more-way tiebreaking criteria for the following reasons: (i) these are far less significant in improving the magic numbers (i.e., moving forward the clinch and elimination dates), and (ii) the computational burden is significant. Specifically, since 2000, which stretches as far back as the penultimate last major lockout, there have been three three-or-more-way ties and one four-way tie, none of which have determined playoff qualification. Therefore, historically, ending the regular season while needing to consider the tiebreaking criteria is rare for a three-or-more-way tie. By contrast, two-way ties are sufficiently prevalent at the end of the season that considering them during the season helps to "tighten" our results, as we show in the following section. Specifically, during the same time frame, there have been 59 two-way ties at the end of the regular season, of which 4 have decided playoff qualification, and considering two-way ties during the season results in improvements to the solution (in terms of moving the date at which one or more magic numbers are determined) for 12 of 30 teams (see Figure 1). Specifically, for the first-place clinch, the playoff clinch, and the elimination number, including the two-way tiebreaking constraints moves the determination up by a day in five cases and by as many as four days in two cases. Note that we cannot compare RIOT's first-place elimination number to the traditional magic number because we have been unable to find a data source that publishes magic-number-based first-place elimination for the NBA. However, authors have established that optimization-based methods can detect first-place elimination earlier than magic number–based methods can (e.g., Robinson 1991).

Regarding the computational burden, we note that the three-or-more-way tiebreaking criteria are difficult to model because, with increasing criterion numbers, the number of teams being compared in the tie changes as teams win or lose the previous tiebreaking criterion. Specifically, Figure 2 contrasts the logic associated with incorporating two-way tiebreaking criteria versus three-or-more-way criteria. Whereas the two-way criteria yield a unidirectional flow of implications, the three-or-more-way criteria loop back on each other, most notably as a function of when and how a three-or-more-way tie is broken. For example, if such a tie is broken after the first criterion into a clear loser and two remaining tied teams, the two tied teams must then visit the two-way tiebreaking criteria loop to determine the winner; however, if three teams remain tied until the fifth three-or-more-way tiebreaking criterion, after which the tie is broken for all three teams, then the loop terminates. Also note that there are separate logical flows based on the number of divisional leaders.

With this in mind, in order to balance model tractability with accuracy, we consider only two-way ties. As a consequence, our results are, at worst, conservative (but not incorrect) regarding elimination dates. Likewise, the elimination numbers posted on RIOT are conservative upper bounds on the number of games a team must win to stay in contention. Conversely, the clinch numbers posted on RIOT are technically lower bounds. In our study of the 2017–2018 season, we detected four playoff clinches earlier than the NBA announced them. In these cases, we verified that the team in question had, in fact, clinched a playoff berth even in the event of a three-or-more-way tie. We describe one such case (the Golden State Warriors) in detail in the *Golden State's Playoff Clinch Date* section.

Data and Results

We use data for the parameter values from the https://www.basketball-reference.com/ website, which provides a list of all the games a team has completed and has yet to play. For the completed games, the winning team and the points scored by both teams are available. These values are constantly changing as the season progresses.

Results

To show the effectiveness of our MIP models, referred to collectively as RIOT, we compare our results against the information posted on the NBA website for the 2017–2018 regular season. Figures 3 and 4 show the number of games remaining in the season and the date on which the respective information source determined when a given team either clinched first place in the conference, clinched a playoff position, or was eliminated from the playoffs for the Eastern and Western Conferences, respectively. Cells highlighted by green vertical lines show RIOT outperforming the NBA's published results, blue horizontal lines represent the case in which the two tie, and red crosshatched lines show the NBA's published results outperforming

Figure 1. (Color online) Comparison of Three of the Four Magic Numbers Based on NBA Information and RIOT With and Without Tiebreaking Constraints

	Eastern Conference (2017-2018)											
	RIOT No Tie						RIOT with Tie					
	Clinch 1st		Clinch Playoffs		Eliminated		Clinch 1st		Clinch Playoffs		Eliminated	
	Date	Games	Date	Games	Date	Games	Date	Games	Date	Games	Date	Games
Atlanta (ATL)	-	-	-	-	3/12	227	-	-	-	-	3/11	231
Boston (BOS)	-	-	3/13	216	-	-	-	-	3/13	216	-	-
Brooklyn (BRN)	-	-	-	-	3/13	216	-	-	-	-	3/13	216
Charlotte (CHA)	-	-	-	-	3/30	93	-	-	-	-	3/30	93
Chicago (CHI)	-	-	-	-	3/19	174	-	-	-	-	3/19	174
Cleveland (CLE)	-	-	3/23	143	-	-	-	-	3/22	153	-	-
Detroit (DET)	-	-	-	-	4/4	56	-	-	-	-	4/4	56
Indiana (IND)	-	-	3/25	128	-	-	-	-	3/25	128	-	-
Miami (MIA)	-	-	4/4	56	-	-	-	-	4/3	62	-	-
Milwaukee (MIL)	-	-	4/4	56	-	-	-	-	4/4	56	-	-
New York (NY)	-	-	-	-	3/21	159	-	-	-	-	3/19	174
Orlando (ORL)	-	-	-	-	3/12	227	-	-	-	-	3/12	227
Philadelphia (PHI)	-	-	3/26	123	-	-	-	-	3/25	128	-	-
Toronto (TOR)	4/8	27	3/7	260	-	-	4/6	40	3/7	260	-	-
Washington (WAS)	-	-	4/4	56	-	-	-	-	3/31	88	-	-

KEY
Tie-break Improves Solution
Tie-break Not Needed

	Western Conference (2017-2018)											
	RIOT No Tie						RIOT with Tie					
	Clinch 1st		Clinch Playoffs		Eliminated		Clinch 1st		Clinch Playoffs		Eliminated	
	Date	Games	Date	Games	Date	Games	Date	Games	Date	Games	Date	Games
Dallas (DAL)	-	-	-	-	3/11	231	-	-	-	-	3/9	245
Denver (DEN)	-	-	-	-	4/11	0	-	-	-	-	4/11	0
Golden State (GSW)	-	-	3/13	216	-	-	-	-	3/9	245	-	-
Houston (HOU)	3/30	93	3/11	231	-	-	3/29	102	3/8	255	-	-
LA Clippers (LAC)	-	-	-	-	4/7	34	-	-	-	-	4/7	34
LA Lakers (LAL)	-	-	-	-	3/26	123	-	-	-	-	3/26	123
Memphis (MEM)	-	-	-	-	3/9	245	-	-	-	-	3/9	245
Minnesota (MIN)	-	-	4/11	0	-	-	-	-	4/11	0	-	-
New Orleans (NO)	-	-	4/9	18	-	-	-	-	4/9	18	-	-
Oklahoma City (OKC)	-	-	4/9	18	-	-	-	-	4/9	18	-	-
Phoenix (PHX)	-	-	-	-	3/7	260	-	-	-	-	3/5	275
Portland (POR)	-	-	4/1	75	-	-	-	-	4/1	75	-	-
Sacramento (SAC)	-	-	-	-	3/11	231	-	-	-	-	3/9	245
San Antonio (SAN)	-	-	4/9	18	-	-	-	-	4/9	18	-	-
Utah (UTH)	-	-	4/8	27	-	-	-	-	4/8	27	-	-

KEY
Tie-break Improves Solution
Tie-break Not Needed

Notes. Dashes denote incompatible information (e.g., if a team has a playoff clinch date, then it lacks a playoff elimination date). ATL, Atlanta Hawks; BOS, Boston Celtics; BRN, Brooklyn Nets; CHA, Charlotte Hornets; CHI, Chicago Bulls; CLE, Cleveland Cavaliers; DAL, Dallas Mavericks; DEN, Denver Nuggets; DET, Detroit Pistons; GSW, Golden State Warriors; HOU, Houston Rockets; IND, Indiana Pacers; MEM, Memphis Grizzlies; MIA, Miami Heat; MIL, Milwaukee Bucks; MIN, Minnesota Timberwolves; NO, New Orleans Pelicans; NY, New York Knicks; OKC, Oklahoma City Thunder; ORL, Orlando Magic; PHI, Philadelphia 76ers; PHX, Phoenix Suns; POR, Portland Trail Blazers; SAC, Sacramento Kings; SAN, San Antonio Spurs; TOR, Toronto Raptors; UTH, Utah Jazz; WAS, Washington Wizards.

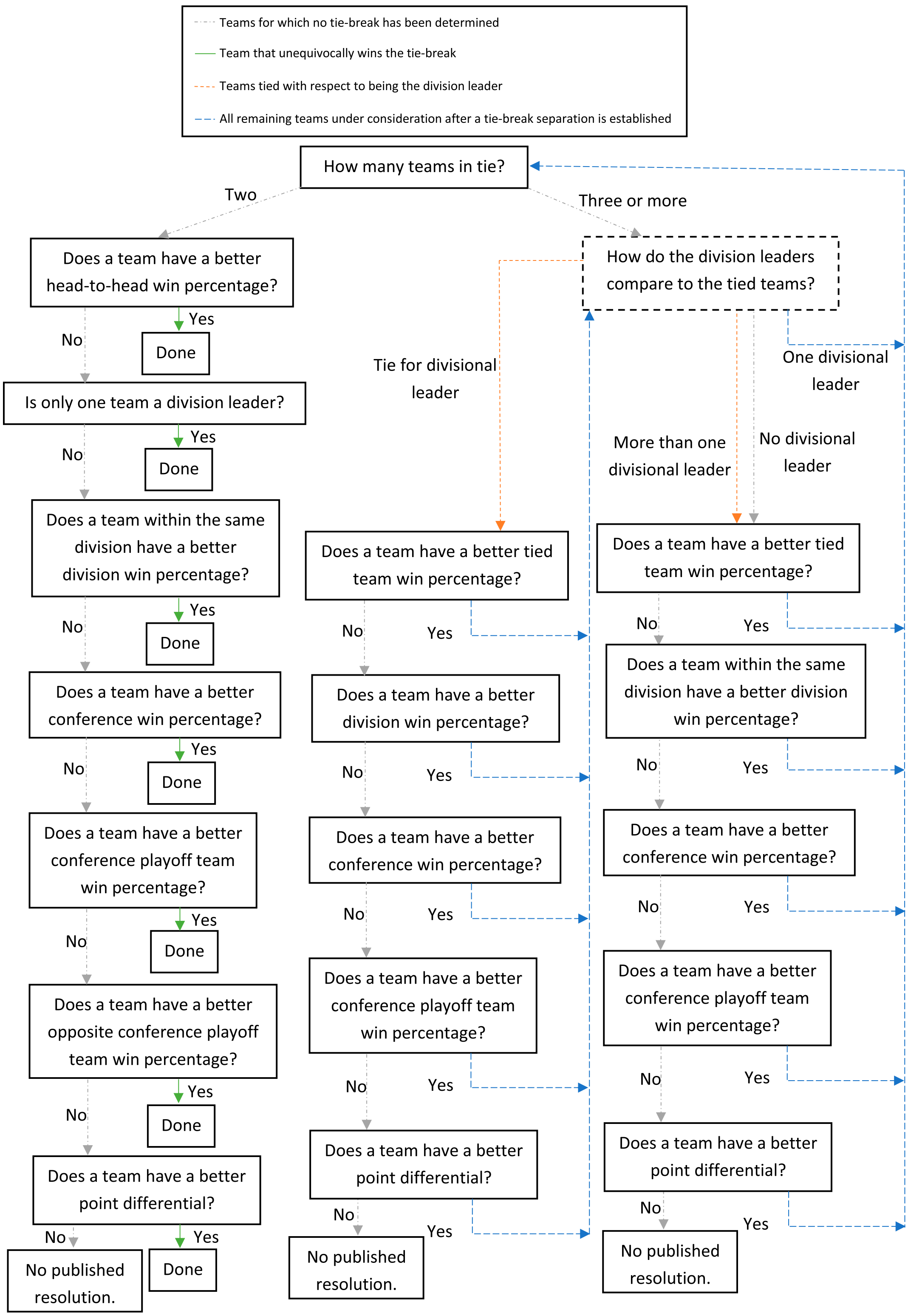
RIOT, where we define outperformance as the ability of a model to determine earlier that a given team had either clinched or been eliminated.

Boston's Playoff Clinch Date

Figure 3 shows that the NBA outperforms RIOT in one instance: namely, that the Boston Celtics clinched a playoff spot on March 8, 2018. However, our playoff clinch model identifies this as erroneous reporting on the part of the NBA. Table 3 gives the records of the top nine teams in the Eastern Conference immediately after all games had been played on this date. Observe that if the Boston Celtics (46 current wins) and Detroit Pistons (29 current wins) were to have lost and won the rest of their games, respectively, the two teams would have both finished the season with 46 wins. This would have put Boston and Detroit in a tie for 8th place because the 10th-place team at the time could have finished the season with at most 45 wins (i.e., the Charlotte Hornets had won 28 games and had 17 left play). Boston had already won the season series with Detroit by beating them two of the three times they played against each other and, hence, the head-to-head tiebreaker (first tiebreaking criterion). So it might appear that Boston had clinched a playoff spot, as announced by the NBA (NBA.com 2018a). Unfortunately for Boston fans, however, this announcement was premature.

Specifically, Table 3 provides a scenario, in terms of the wins and losses for the top nine teams in the Eastern Conference, in which Boston would not make the playoffs based on the standings and schedule of remaining games as of the end of the night on

Figure 2. (Color online) The Left-Hand Side of the Flowchart Depicts the Linear Logical Flow of the Two-Way Tiebreaking Criteria, Whereas the Right-Hand Side Demonstrates the Complexities (in Terms of Loops) Associated with Three-or-More-Way Ties



March 8, 2018. In our scenario, the Toronto Raptors win the Atlantic Division, the Washington Wizards win the Southeast Division, and the Milwaukee Bucks win the Central Division; the Philadelphia 76ers have enough wins to qualify for the playoffs, and there is a five-way tie for the last four playoff positions between

Figure 3. (Color online) The Eastern Conference Comparison of the Date ("Date") and Number of Games Remaining ("Games") When a Particular Result Was Determined (i.e., First-Place Clinch, Playoff Clinch, and Elimination) Based on the NBA's Computation and RIOT

Eastern Conference (2017-2018)												
	NBA						RIOT					
	Clinch 1st		Clinch Playoffs		Eliminated		Clinch 1st		Clinch Playoffs		Eliminated	
	Date	Games	Date	Games	Date	Games	Date	Games	Date	Games	Date	Games
Atlanta (ATL)	-	-	-	-	3/12	227	-	-	-	-	3/11	231
Boston (BOS)	-	-	3/8	255	-	-	-	-	3/13	216	-	-
Brooklyn (BRN)	-	-	-	-	3/14	212	-	-	-	-	3/13	216
Charlotte (CHA)	-	-	-	-	3/30	93	-	-	-	-	3/30	93
Chicago (CHI)	-	-	-	-	3/20	167	-	-	-	-	3/19	174
Cleveland (CLE)	-	-	3/23	143	-	-	-	-	3/22	153	-	-
Detroit (DET)	-	-	-	-	4/4	56	-	-	-	-	4/4	56
Indiana (IND)	-	-	3/25	128	-	-	-	-	3/25	128	-	-
Miami (MIA)	-	-	4/3	62	-	-	-	-	4/3	62	-	-
Milwaukee (MIL)	-	-	4/4	56	-	-	-	-	4/4	56	-	-
New York (NY)	-	-	-	-	3/20	167	-	-	-	-	3/19	174
Orlando (ORL)	-	-	-	-	3/14	212	-	-	-	-	3/12	227
Philadelphia (PHI)	-	-	3/25	128	-	-	-	-	3/25	128	-	-
Toronto (TOR)	4/6	40	3/8	255	-	-	4/6	40	3/7	260	-	-
Washington (WAS)	-	-	3/31	88	-	-	-	-	3/31	88	-	-

KEY
RIOT beats NBA
RIOT ties NBA
NBA beats RIOT

the Boston Celtics, Cleveland Cavaliers, Detroit Pistons, Indiana Pacers, and Miami Heat. Because none of the tied teams is leading a division, the second criterion breaks the five-way tie. Table 4 shows the actual outcomes of the games between these five teams played prior to March 8, 2018, and the following scenarios: Indiana beats Boston and Miami, and Miami beats Cleveland. In this case, Boston would have the worst tied team win percentage and be eliminated from the playoffs. In actual fact, Boston did clinch a playoff spot, but not until five days later, as determined by our RIOT model and shown in red crosshatched lines in Figure 3. In Appendix E, we explain how we used the RIOT model to find the scenario described in Table 3.

Golden State's Playoff Clinch Date

In this section, we show that Golden State had clinched a playoff spot on March 9, 2018, three days before the official announcement by the NBA (NBA.com 2018b). That is, we show that there was no scenario in which Golden State would not have made the playoffs (i.e., we explain why their playoff clinch mixed-integer program was infeasible). Table 5 shows the Western Conference standings at the time in question. The "Maximum possible wins" column shows that the five teams in the out-of-contention group were going to finish the season behind Golden State, which had already won 51 games. It appears that Golden State had not yet clinched a playoff spot because there were nine teams that could have potentially finished with better records: the Houston Rockets and the eight teams in the in-contention group. However, it turns out that regardless of the way in which the rest of the games in the season concluded, Golden State was guaranteed to make the playoffs.

Using a straightforward calculation, at most seven of the teams in the in-contention group could have finished the season with 51 or more wins, which

Figure 4. (Color online) The Western Conference Comparison of the Date ("Date") and Number of Games Remaining ("Games") When a Particular Result Was Determined (i.e., First-Place Clinch, Playoff Clinch, and Elimination) Based on the NBA's Computation and RIOT

Western Conference (2017-2018)												
	NBA						RIOT					
	Clinch 1st		Clinch Playoffs		Eliminated		Clinch 1st		Clinch Playoffs		Eliminated	
	Date	Games	Date	Games	Date	Games	Date	Games	Date	Games	Date	Games
Dallas (DAL)	-	-	-	-	3/10	240	-	-	-	-	3/9	245
Denver (DEN)	-	-	-	-	4/11	0	-	-	-	-	4/11	0
Golden State (GSW)	-	-	3/13	216	-	-	-	-	3/9	245	-	-
Houston (HOU)	3/29	102	3/12	227	-	-	3/29	102	3/8	255	-	-
LA Clippers (LAC)	-	-	-	-	4/7	34	-	-	-	-	4/7	34
LA Lakers (LAL)	-	-	-	-	3/27	115	-	-	-	-	3/26	123
Memphis (MEM)	-	-	-	-	3/10	240	-	-	-	-	3/9	245
Minnesota (MIN)	-	-	4/11	0	-	-	-	-	4/11	0	-	-
New Orleans (NO)	-	-	4/9	18	-	-	-	-	4/9	18	-	-
Oklahoma City (OKC)	-	-	4/9	18	-	-	-	-	4/9	18	-	-
Phoenix (PHX)	-	-	-	-	3/9	245	-	-	-	-	3/5	275
Portland (POR)	-	-	4/1	75	-	-	-	-	4/1	75	-	-
Sacramento (SAC)	-	-	-	-	3/10	240	-	-	-	-	3/9	245
San Antonio (SAN)	-	-	4/9	18	-	-	-	-	4/9	18	-	-
Utah (UTH)	-	-	4/8	27	-	-	-	-	4/8	27	-	-

KEY
RIOT beats NBA
RIOT ties NBA

Table 3. A Scenario in Which Boston Does Not Make the 2018 Playoffs as of March 8, 2018

Team	Current record	Hypothetical results of remaining games for the top nine teams											Final record
TOR	47–17	Wins	BOS 2	IND 2									51–31
		Losses	BRN 2	CLE 2	DAL	DEN	DET	HOU	LAC	MIA	NY	OKC	
			ORL 2										
WAS	37—28	Wins	BOS 2	CHA	CHI	DEN	HOU	IND	MIA	MIN	NO	NY	51–31
			ORL	SAN 2									
		Losses	ATL	CLE	DET								
MIL	34–31	Wins	BOS	BRN	CHI	CLE	DEN	GSW	LAL	MEM	NY 2	ORL 2	47–35
			SAN										
		Losses	ATL	LAC 2	PHI								
PHI	35–29	Wins	BRN 3	CHA 2	DAL	DEN	MIL	MIN	NY 2	ORL			47–35
		Losses	ATL 2	CLE	DET	IND	MEM						
CLE	38–26	Wins	BRN	LAL	NY 2	PHI	TOR 2	WAS					46–36
		Losses	CHA	CHI	DAL	LAC	MIA	MIL	NO	PHX 2	POR		
DET	29–36	Wins	BRN	CHI 3	DAL	DEN	HOU	LAL	MEM	NY	PHI	PHX	46–36
			POR	SAC	TOR	UTH	WAS						
		Losses											
IND	37–28	Wins	BOS	CHA 2	DEN	GSW 2	LAL	MIA	PHI				46–36
		Losses	ATL	LAC 2	NO	SAC	TOR 2	WAS					
MIA	35–31	Wins	CHI	CLE	DEN	LAL	NY 2	OKC 2	POR	SAC	TOR		46–36
		Losses	ATL 2	BRN	IND	WAS							
BOS	46–20	Wins											46–36
		Losses	ATL	BRN	CHI	IND	MIL	NO	OKC	ORL	PHX	POR	
			SAC	TOR 2	UTH	WAS 2							

means that Golden State was already guaranteed to finish the season ahead of at least six teams in the conference. The sum of the maximum possible wins for this group was 428, which counts the games in the rightmost column of Table 5 twice; that is, a game between team k and team k' is counted in both teams' rows. Therefore, the maximum collective win total for this group was $428 - \left(\frac{50}{2}\right) = 403$, and the maximum average number of wins for these teams was $\frac{403}{8} = 50.38$. This implies that at least one of these eight teams would necessarily finish the season with $\lfloor 50.38 \rfloor = 50$ or fewer wins (i.e., behind Golden State in the standings), because it is impossible to win a fractional number of games.

Using this observation, we can significantly reduce the number of scenarios under consideration. For example, suppose the Utah Jazz are the team with 50 or fewer wins. In this case, the data in Table 5 indicate that the seven other teams in the in-contention group would have a maximum collective win total equal to the maximum collective win total for the eight teams, less the possible wins for Utah (which do not count into the total of the seven other teams), less the games against the in-contention teams, assuming Utah loses all of these. This calculation is $403 - (52 - 5) = 356$ and yields a maximum average for the seven other teams of $\frac{356}{7} = 50.86$ wins per team. Again, because teams cannot win fractional games, at least one team in the group of seven would have a win record of $\lfloor 50.86 \rfloor = 50$, signifying that at least one team besides Utah would have to finish the season with 50 or fewer wins. This implies that Golden State would be guaranteed to

Table 4. The Tied Team Win Percentages for Teams with a Final Season Record of 46–36, Where the Sum of Actual Wins and Scenario Wins Comprise the Total Team Wins

	Wins against							
Team	Boston	Cleveland	Detroit	Indiana	Miami	Wins	Losses	Percentage
Boston	—	1	2	2	1	6	7	0.462
Cleveland	2	—	3	1	2	8	6	0.571
Detroit	1	1	—	3	2	7	8	0.467
Indiana	1 + 1	3	1	—	1 + 1	8	8	0.500
Miami	2	0 + 1	2	2	—	7	7	0.500

Table 5. The Western Conference Standings After All Games Have Been Played on March 9, 2018

	Team	Division	Wins	Losses	Maximum possible wins	Games remaining against other teams in contention
Past contention	Houston Rockets	Southwest	51	14	68	—
	Golden State Warriors	Pacific	51	15	67	—
In contention	Portland Trail Blazers	Northwest	40	26	56	7
	New Orleans Pelicans	Southwest	38	27	55	6
	San Antonio Spurs	Southwest	37	28	54	8
	Minnesota Timberwolves	Northwest	38	29	53	5
	Oklahoma City Thunder	Northwest	38	29	53	6
	LA Clippers	Pacific	35	29	53	8
	Denver Nuggets	Northwest	36	30	52	5
	Utah Jazz	Northwest	36	30	52	5
Out of contention	LA Lakers	Pacific	29	36	46	—
	Sacramento Kings	Pacific	21	45	37	—
	Dallas Mavericks	Southwest	20	45	37	—
	Phoenix Suns	Pacific	19	48	34	—
	Memphis Grizzlies	Southwest	18	47	35	—

finish in a playoff position ahead of at least seven teams in the conference: two in the in-contention group and five in the out-of-contention group.

However, up to this point, we have only shown that Golden State would be guaranteed to finish in a playoff position were Utah to be the team with 50 or fewer wins. Table 6 shows the maximum average number of wins achievable by all teams in the in-contention group in any scenario in which a particular team from the group finishes behind Golden State in the standings with fewer than 51 wins and misses the playoffs. As in the Utah case, Golden State makes the playoffs in all the cases for which the average is 50. The table shows that there are only two cases in which Golden State could conceivably not make the playoffs as a result of being tied in the standings with seven 51-win teams in the in-contention group:

1. *The Los Angeles (LA) Clippers finish the season with fewer than 51 wins:* In this case, Golden State would have more wins than any of the other teams in the Pacific Division and therefore make the playoffs by winning the tiebreaker.

2. *The San Antonio Spurs finish the season with fewer than 51 wins:* In this case, Golden State could finish in a 51-win tie with its division rival, the LA Clippers. However, Golden State would still be declared the winner of the Pacific Division because it had already won the season series against the LA Clippers, three games to one.

Thus, Golden State had clinched a playoff spot on March 9, 2018. For the sake of brevity, we omit a similar logical argument demonstrating that Houston had also clinched a playoff spot on that date.

RIOT Website

The RIOT website implements the mixed-integer programs in the AMPL modeling language (AMPL 2009) and solves them with the CPLEX Solver, version 12.6.0.0 (IBM 2011), on a Dell R720 with 266 GB of memory and Dual six-core Intel Xeon processors running at 3.5 GHz under the CentOS Linux (release 6.10) operating system. The time required to solve each of the models for all 30 NBA teams varies considerably throughout the season, from an average of 12 hours for all four models and 30 teams, in which there are over $2^{1,000}$ scenarios in the first few days of the season, to the final days of the season, in which the four models for all 30 teams collectively require less

Table 6. The Maximum Average Number of Wins for the In-Contention Group (See Table 5)

Team missing the playoffs	Maximum average number of wins among the remaining seven teams
None	$\lfloor 50.38 \rfloor$
Denver Nuggets	$\lfloor 50.86 \rfloor$
LA Clippers	$\lfloor 51.14 \rfloor$
Minnesota Timberwolves	$\lfloor 50.71 \rfloor$
New Orleans Pelicans	$\lfloor 50.57 \rfloor$
Oklahoma City Thunder	$\lfloor 50.86 \rfloor$
Portland Trail Blazers	$\lfloor 50.57 \rfloor$
San Antonio Spurs	$\lfloor 51.00 \rfloor$
Utah Jazz	$\lfloor 50.86 \rfloor$

than a minute; midseason, solve times average between 4 and 6 hours.

Conclusions

We incorporate tiebreaking criteria into mixed-integer linear optimization models to determine when NBA teams have clinched a playoff spot or have been eliminated from the playoffs. In several cases, our mathematical models enable us to determine these results sooner than when they are published on the NBA official website. We also are able to show that the NBA erroneously announced the dates on which the Golden State Warriors and Boston Celtics clinched their playoff spots in 2018. However, our process is not fully automated. Because we do not model the complex procedure for resolving ties between three or more teams as a mathematical program, we use a manual verification process when the RIOT models determine a clinch or an elimination. This helps to ensure that our models neither (i) prematurely eliminate a team nor (ii) "guarantee" that a given team has clinched a playoff spot when, in fact, it has not. Future work could incorporate the three-or-more-way tiebreaking criteria and tighten the constraints by using the two-way, seventh criterion. Since 2000, there have been only four occurrences of an end-of-season three-or-more-way tie, none of which determined the teams that made the playoffs (although they did determine playoff seeding), granting a high degree of confidence in the RIOT numbers. Nevertheless, modeling the procedure for breaking three-or-more-way ties would allow the RIOT website to post exact magic numbers (rather than bounds) for the NBA as it does for Major League Baseball.

The majority of the playoff teams in the National Collegiate Athletic Association's (NCAA) men's basketball tournament are determined by a selection committee. The process is subjective and would seem to fall outside the scope of the methodology developed here and described in the literature review. However, Reinig and Horowitz (2018) propose an optimization model to select, rank, and seed the postseason college basketball tournament. Their model produces results that closely approximate the actual selections made in the 2012–2017 tournaments. Determining clinch and elimination numbers for NCAA tournament selection based on the results of Reinig and Horowitz (2018) would be an interesting direction for future work.

Acknowledgments

The authors thank Nelson "Trey" Winbush for his help in data collection and two anonymous referees for their comments on earlier drafts of the paper.

Appendix A. Mathematical Formulation for Playoff Elimination

The following formulation is an integer program for a given team k to determine the number of wins needed to preclude its elimination from the playoffs. If the model is infeasible, then there is no scenario in which team k can qualify for the playoffs; that is, team k has been eliminated from the playoffs. This mixed-integer program, $(\mathcal{M}_k)$, is run for each team k. The notation is provided in Table A.1.

Table A.1. Notation

Notation	Definition
	Sets
$\mathcal{T}$	All teams in the league
$\mathcal{C}_t$	Teams within the same conference as team t, excluding team t itself
$\mathcal{D}_t$	Teams within the same division as team t
$\mathcal{R}$	Criteria used to break a two-way tie, ordered by decreasing importance
	Playoff elimination parameters
$\hat{c}$	Number of conference games each team plays
$\hat{d}$	Number of divisional games each team plays
$g_{tt'}$	Number of games left between team t and team t'
$h_{tt'}$	Current number of wins team t has against team t'
k	Team in question
M	Sufficiently large number
n	Number of teams that make the playoffs from each conference
$\hat{n}$	Number of teams in a division
$\hat{w}_t$	Current number of wins for team t
	Binary, continuous, and integer variables
Continuous variables	
X_t^C	Conference win percentage of team t in the scenario
X_t^D	Division win percentage of team t in the scenario
X_t^{OC}	Other conference win percentage of team t for all playoff teams in the scenario
X_t^{PC}	Conference win percentage of team t for all playoff teams in the scenario
X_t^T	Win percentage of team t between all tied teams in the scenario
Integer variables	
$Y_{tt'}$	Wins team t has over team t' in the scenario
W_t	Total number of games team t wins in the scenario
Binary variables	
$\alpha_{tt'}$	Equal to 1 if team t wins the tie-breaker over team t' in the scenario and 0 otherwise
$\beta_{tt'}$	Equal to 1 if team t has more wins than team t' ($W_t > W_{t'}$) in the scenario and 0 otherwise
ω_{tk}	Equal to 1 if team t is listed higher in the standings than team k in the scenario and 0 otherwise
$\kappa_{tt'}$	Equal to 1 if team t has more wins in a divisional tie than team t' in the scenario and 0 otherwise
γ_t	Equal to 1 if team t is either in the playoffs or tied for being in the playoffs in the scenario and 0 otherwise
λ	Equal to 1 if the team in question is tied for the last playoff position and 0 otherwise
$T_{tt'r}$	Equal to 1 if team t wins tiebreaking criterion r over team t' in the scenario and 0 otherwise
T_t^d	Equal to 1 if team t wins its division in the scenario and 0 otherwise
$Z_{tt'}$	Equal to 1 if there is a tie between team t and team t' in the scenario and 0 otherwise

Problem $(\mathcal{M}_k)$
(See the Objective Function section.)

$$\text{Minimize } W_k \tag{A.1}$$

subject to (see the Win Comparison section)

$$Y_{tt'} + Y_{t't} = g_{tt'} \quad \forall t, t' \in \mathcal{T} \tag{A.2a}$$
$$W_t = \hat{w}_t + \sum_{t' \in \mathcal{T}} Y_{tt'} \quad \forall t \in \mathcal{T} \tag{A.2b}$$
$$W_k + M \cdot \omega_{tk} \geq W_t + \frac{\alpha_{tk}}{2} \quad \forall t \in \mathcal{C}_k \tag{A.2c}$$
$$\sum_{t \in \mathcal{C}_k} \omega_{tk} \leq n - 1 \tag{A.2d}$$

(see the Determination of Ties section)

$$\alpha_{tt'} + \alpha_{t't} = Z_{tt'} \quad \forall t, t' \in \mathcal{C}_k \tag{A.3a}$$
$$Z_{tt'} = Z_{t't} \quad \forall t, t' \in \mathcal{C}_k \tag{A.3b}$$
$$T_{tt'r} + T_{t'tr} \leq 1 \quad \forall t, t' \in \mathcal{C}_k, r \in \mathcal{R} \tag{A.3c}$$
$$W_t - W_{t'} \leq M \cdot (1 - Z_{tt'}) \quad \forall t, t' \in \mathcal{C}_k \tag{A.3d}$$
$$\frac{1}{2} - (W_t - W_{t'}) \leq Z_{tt'} + M \cdot \beta_{t't} \quad \forall t, t' \in \mathcal{C}_k \tag{A.3e}$$
$$\frac{1}{2} - (W_{t'} - W_t) \leq Z_{tt'} + M \cdot (1 - \beta_{t't}) \quad \forall t, t' \in \mathcal{C}_k \tag{A.3f}$$
$$Z_{tt'} + \beta_{tt'} + \beta_{t't} = 1 \quad \forall t, t' \in \mathcal{C}_k \tag{A.3g}$$

(see the Other Conference Playoff Teams section)

$$W_t + M \cdot \beta_{t't} + \frac{1}{2} \geq W_{t'} \quad \forall t, t' \notin \mathcal{C}_k \tag{A.4a}$$
$$W_t - M \cdot (1 - \beta_{t't}) + \frac{1}{2} \leq W_{t'} \quad \forall t, t' \notin \mathcal{C}_k \tag{A.4b}$$
$$M \cdot \gamma_t \geq \left(n - \frac{1}{2}\right) - \sum_{t' \in \mathcal{C}_t} \beta_{t't} \quad \forall t \in \mathcal{T} \tag{A.4c}$$
$$M \cdot (1 - \gamma_t) \geq \sum_{t' \in \mathcal{C}_t} \beta_{t't} - \left(n - \frac{1}{2}\right) \quad \forall t \in \mathcal{T} \tag{A.4d}$$

(see the Win Percentage section)

$$X_t^T = \frac{\sum_{t' \in \mathcal{C}_k} [(h_{tt'} + Y_{tt'}) \cdot Z_{tt'}]}{\sum_{t' \in \mathcal{C}_k} [(h_{tt'} + h_{t't} + g_{tt'}) \cdot Z_{tt'}]} \quad \forall t \in \mathcal{C}_k \tag{A.5a}$$
$$X_t^T \leq \sum_{t' \in \mathcal{C}_k} Z_{tt'} \quad \forall t \in \mathcal{C}_k \tag{A.5b}$$
$$X_t^D = \frac{\sum_{t' \in \mathcal{D}_t} (h_{tt'} + Y_{tt'})}{\hat{d}} \quad \forall t \in \mathcal{C}_k \tag{A.5c}$$
$$X_t^C = \frac{\sum_{t' \in \mathcal{C}_k} (h_{tt'} + Y_{tt'})}{\hat{c}} \quad \forall t \in \mathcal{C}_k \tag{A.5d}$$
$$X_t^{PC} = \frac{\sum_{t' \in \mathcal{C}_k} [(h_{tt'} + Y_{tt'}) \cdot \gamma_{t'}]}{\sum_{t' \in \mathcal{C}_k} [(h_{tt'} + h_{t't} + g_{tt'}) \cdot \gamma_{t'}]} \quad \forall t \in \mathcal{C}_k \tag{A.5e}$$
$$X_t^{OC} = \frac{\sum_{t' \notin \mathcal{C}_k} [(h_{tt'} + Y_{tt'}) \cdot \gamma_{t'}]}{\sum_{t' \notin \mathcal{C}_k} [(h_{tt'} + h_{t't} + g_{tt'}) \cdot \gamma_{t'}]} \quad \forall t \in \mathcal{C}_k \tag{A.5f}$$

(see the Tie-breaker Criterion 1 section)

$$\frac{1}{M} - \left(1 + \frac{1}{M}\right)(1 - T_{tt',1}) \leq X_t^T - X_{t'}^T \leq T_{tt',1} \quad \forall t, t' \in \mathcal{C}_k \tag{A.6}$$

(see the Tie-breaker Criterion 2 section)

$$\sum_{t' \in \mathcal{D}_t} (\beta_{tt'} + (\kappa_{tt'} \cdot Z_{tt'})) \leq (\hat{n} - 2) + T_t^d \quad \forall t \in \mathcal{C}_k \tag{A.7a}$$
$$\sum_{t' \in \mathcal{D}_t} (\beta_{tt'} + (\kappa_{tt'} \cdot Z_{tt'})) \geq (\hat{n} - 1) \cdot T_t^d \quad \forall t \in \mathcal{C}_k \tag{A.7b}$$
$$2 \cdot T_{tt',2} - 1 \leq T_t^d - T_{t'}^d \leq T_{tt',2} \quad \forall t, t' \in \mathcal{C}_k \tag{A.7c}$$
$$Y_{tt'} + h_{tt'} - Y_{t't} - h_{t't} \leq M \cdot \kappa_{tt'} \quad \forall t \in \mathcal{C}_k, t' \in \mathcal{D}_t \tag{A.7d}$$
$$\frac{1}{2} - M \cdot (1 - \kappa_{tt'}) \leq Y_{tt'} + h_{tt'} - Y_{t't} - h_{t't} \quad \forall t \in \mathcal{C}_k, t' \in \mathcal{D}_t \tag{A.7e}$$

(see the Tie-breaker Criterion 3 section)

$$\frac{1}{M} - \left(1 + \frac{1}{M}\right)(1 - T_{tt',3}) \leq X_t^D - X_{t'}^D \leq T_{tt',3} \quad \forall t \in \mathcal{C}_k, t' \in \mathcal{D}_t \tag{A.8a}$$
$$T_{tt',3} = 0 \quad \forall t \in \mathcal{C}_k, t' \notin \mathcal{D}_t \tag{A.8b}$$

(see the Tie-breaker Criterion 4 section)

$$\frac{1}{M} - \left(1 + \frac{1}{M}\right)(1 - T_{tt',4}) \leq X_t^C - X_{t'}^C \leq T_{tt',4} \quad \forall t, t' \in \mathcal{C}_k \tag{A.9}$$

(see the Tie-breaker Criterion 5 section)

$$\frac{1}{M} - \left(1 + \frac{1}{M}\right)(1 - T_{tt',5}) \leq X_t^{PC} - X_{t'}^{PC} \leq T_{tt',5} \quad \forall t, t' \in \mathcal{C}_k \tag{A.10}$$

(see the Tie-breaker Criterion 6 section)

$$\frac{1}{M} - \left(1 + \frac{1}{M}\right)(1 - T_{tt',6}) \leq X_t^{OC} - X_{t'}^{OC} \leq T_{tt',6} \quad \forall t, t' \in \mathcal{C}_k \tag{A.11}$$

(see the Tie-breaker Winner section)

$$2^{n+1} \cdot (1 - Z_{tt'}) + \sum_{r \in \mathcal{R}} 2^{n-r} \cdot T_{tt'r} + 2^n \cdot \alpha_{t't} \geq \sum_{r \in \mathcal{R}} 2^{n-r} \cdot T_{t'tr} \quad \forall t, t' \in \mathcal{C}_k \tag{A.12}$$

(see the Exclusion of Multiteam Ties section)

$$\sum_{t \in \mathcal{C}_k} Z_{kt} \leq |\mathcal{C}_k| \cdot (1 - \lambda) + \lambda \tag{A.13a}$$
$$\sum_{t \in \mathcal{C}_k} \beta_{kt} \geq (|\mathcal{C}_k| + 1 - n) \cdot (1 - \lambda) \tag{A.13b}$$

(see the Nonnegativity and Integrality section)

$$X_t^C, X_t^D, X_t^{OC}, X_t^{PC}, X_t^T \geq 0 \quad \forall t \in \mathcal{C}_k \tag{A.14a}$$
$$W_t \geq 0 \quad \text{integer} \quad \forall t \in \mathcal{T} \tag{A.14b}$$
$$Y_{tt'} \geq 0 \quad \text{integer} \quad \forall t, t' \in \mathcal{T} \tag{A.14c}$$
$$T_t^d \quad \text{binary} \quad \forall t \in \mathcal{C}_k \tag{A.14d}$$
$$\gamma_t \quad \text{binary} \quad \forall t \notin \mathcal{C}_k \tag{A.14e}$$

$$\alpha_{tt'}, \omega_{tt'}, Z_{tt'} \quad \text{binary} \quad \forall t, t' \in \mathscr{C}_k \tag{A.14f}$$
$$\beta_{tt'} \quad \text{binary} \quad \forall t, t' \in \mathscr{T} \tag{A.14g}$$
$$\kappa_{tt'} \quad \text{binary} \quad \forall t \in \mathscr{C}_k, t' \in \mathscr{D}_t \tag{A.14h}$$
$$T_{tt'r} \quad \text{binary} \quad \forall t, t' \in \mathscr{C}_k, r \in \mathscr{R} \tag{A.14i}$$
$$\lambda \quad \text{binary} \tag{A.14j}$$

Objective Function. The objective function (A.1) minimizes the number of wins needed for team k to preclude its elimination from the playoffs. If the model is infeasible, then there is no scenario in which team k can qualify for the playoffs, and correspondingly, team k has been eliminated from the playoffs.

Constraints

Win Comparison. Constraint (A.2a) ensures that each of the remaining games between team t and team t' is won by either of the two teams. Constraint (A.2b) calculates the total number of wins for team t as the number of games team t has already won and the remaining number of games team t wins. Constraint (A.2c) indicates whether team t has a better record than team k (in which case $\omega_{tk} = 1$). Constraint (A.2d) limits the number of teams with more wins than team k to be one fewer than the number of teams that make the playoffs. That is, team k is eliminated from the playoffs if n or more teams have better records.

Determination of Ties. These constraints measure relative performance for two teams in the same conference as the team in question, k. Constraint (A.3a) ensures that either team t or team t' wins the tie-breaker. Constraint (A.3b) guarantees the irrelevance in the order of the indices between tied teams. Constraint (A.3c) ensures that at most one team can win a tiebreaking criterion r. Constraints (A.3d)–(A.3f) ensure that the variable $Z_{tt'}$, representing whether there is a tie between teams t and t', is 1 if there is a tie and 0 otherwise by measuring the difference between the number of wins associated with each team. Constraint (A.3g) ensures that there is a tie or that either team t or team t' has more wins. Variable $\beta_{tt'}$ controls the relevance of Constraints (A.3e) and (A.3f) via a big-M construct.

Other Conference Playoff Teams. These constraints measure relative performance for two teams not in the same conference as team k. Constraints (A.4a) and (A.4b) indicate whether team t' has a better record than team t for all teams t and t' not in the same conference as team k. Constraints (A.4c) and (A.4d) limit the number of teams with more wins than team t to one fewer than the number of teams that make the playoffs.

Win Percentage. The win percentages are used to determine whether a team wins a tiebreaking criterion and are used for the tie-breaker criteria constraints. Constraint (A.5a) determines team t's win percentage against tied teams. Constraint (A.5b) ensures that team t does not have a tied team win percentage if team t is not tied with any teams. Constraint (A.5c) determines team t's division win percentage. Constraint (A.5d) determines team t's conference win percentage. Constraint (A.5e) determines team t's conference win percentage between teams in the conference that have already made the playoffs. In contrast to Constraints (A.5d) and (A.5e), Constraint (A.5f) determines a team's win percentage between all playoff teams not in team k's conference.

Tie-breaker Criterion 1. Constraint (A.6) determines whether a team wins the tie-breaker criterion by having the best win percentage among all tied teams.

Tie-breaker Criterion 2. Constraints (A.7a) and (A.7b) determine whether team t wins its division by having a better overall win percentage than those of the other four teams in its division. If the two teams are tied in overall wins, then κ is used if a team wins the head-to-head series. Constraint (A.7c) determines whether a team wins the tie-breaker criterion by being the only division leader. Constraints (A.7d) and (A.7e) determine whether team t has more head-to-head wins against team t'.

Tie-breaker Criterion 3. Constraint (A.8a) determines whether team t wins the tie-breaker criterion by having a better win percentage in its division than a team t' with which it is tied. Constraint (A.8b) does not allow team t to win the tie-breaker if the team with which it is tied, t', is not in the same division.

Tie-breaker Criterion 4. Constraint (A.9) determines whether team t wins the tie-breaker criterion by having a better win percentage in its conference.

Tie-breaker Criterion 5. Constraint (A.10) determines whether team t wins the tie-breaker criterion by having a better win percentage against all conference playoff teams.

Tie-breaker Criterion 6. Constraint (A.11) determines whether team t wins the tie-breaker criterion by having a better win percentage against all teams in the other conference that have qualified for the playoffs.

Tie-breaker Winner. Constraint (A.12) indicates which team wins the tie-breaker and qualifies for the playoffs.

Exclusion of Multiteam Ties. Constraints (A.13a) and (A.13b) ensure that team k ties with no more than a single team in its conference. For the case in which $\lambda = 1$, team k is tied with at most one other team (by (A.13a)) and wins the tiebreaker against that team, if applicable (based on other constraints in the model). In this case, Constraint (A.13a) is active and Constraint (A.13b) is void. On the other hand, if $\lambda = 0$, Constraint (A.13b) ensures that team k has more wins than at least $|\mathscr{C}_k| + 1 - n$ other teams in the conference, which guarantees that team k is a playoff team, regardless of ties. Because ties are irrelevant in this case, Constraint (A.13a) is void; in other words, team k could be tied with any number of other teams in the conference. Because we do not model the complex process for resolving three-or-more-way ties for playoff spots in the *elimination* models, we restrict them to scenarios in which team k is tied with at most one other team. This ensures that our elimination numbers represent valid upper bounds on the number of games a team must win to avoid elimination.

Nonnegativity and Integrality. Finally, Constraints (A.14a)–(A.14c) ensure that the appropriate variables assume continuous or integer, nonnegative values. Constraints (A.14d)–(A.14j) enforce binary restrictions.

In what follows in Appendices B and C, we detail how to make model ($\mathcal{M}_k$) more tractable. These performance-enhancing techniques can be modified and applied to the models we introduce in Appendix D. Further solution-time reductions may be necessary for particularly difficult-to-solve instances that occur at the beginning of the season when the model is highly underconstrained. We mention these techniques on the RIOT website, https://s2.smu.edu/~olinick/riot/basketball_main.html.

Appendix B. Linearization

In ($\mathcal{M}_k$), there are seven nonlinear terms within Constraints (A.5a), (A.5e), (A.5f), (A.7a), and (A.7b). In what follows, we provide an exact method to linearize these bilinear relationships.

Multiplying the denominator over to the left-hand side, we obtain

$$X_t^T \cdot \sum_{t' \in \mathcal{C}_k} \left[\left(h_{tt'} + h_{t't} + g_{tt'} \right) \cdot Z_{tt'} \right] = \sum_{t' \in \mathcal{C}_k} \left[\left(h_{tt'} + Y_{tt'} \right) \cdot Z_{tt'} \right] \quad \forall t \in \mathcal{C}_k \tag{B.1a}$$

$$X_t^{PC} \cdot \sum_{t' \in \mathcal{C}_k} \left[\left(h_{tt'} + h_{t't} + g_{tt'} \right) \cdot \gamma_{t'} \right] = \sum_{t' \in \mathcal{C}_k} \left[\left(h_{tt'} + Y_{tt'} \right) \cdot \gamma_{t'} \right] \quad \forall t \in \mathcal{C}_k \tag{B.1b}$$

$$X_t^{OC} \cdot \sum_{t' \notin \mathcal{C}_k} \left[\left(h_{tt'} + h_{t't} + g_{tt'} \right) \cdot \gamma_{t'} \right] = \sum_{t' \notin \mathcal{C}_k} \left[\left(h_{tt'} + Y_{tt'} \right) \cdot \gamma_{t'} \right] \quad \forall t \in \mathcal{C}_k \tag{B.1c}$$

$$\sum_{t' \in \mathcal{D}_t} \left(\beta_{tt'} + \left(\kappa_{tt'} \cdot Z_{tt'} \right) \right) \le (\hat{n} - 2) + T_t^d \quad \forall t \in \mathcal{C}_k \tag{B.1d}$$

$$\sum_{t' \in \mathcal{D}_t} \left(\beta_{tt'} + \left(\kappa_{tt'} \cdot Z_{tt'} \right) \right) \ge (\hat{n} - 1) \cdot T_t^d \quad \forall t \in \mathcal{C}_k \tag{B.1e}$$

Within Equations (B.1a)–(B.1e), there are seven bilinear terms, six of which are distinct. We define a variable for each bilinear term in Equations (B.2a)–(B.2f):

$$V_{1tt'} = X_t^T Z_{tt'} \quad \forall t, t' \in \mathcal{C}_k \tag{B.2a}$$
$$V_{2tt'} = Y_{tt'} Z_{tt'} \quad \forall t, t' \in \mathcal{C}_k \tag{B.2b}$$
$$V_{3tt'} = X_t^{PC} \gamma_{t'} \quad \forall t, t' \in \mathcal{C}_k \tag{B.2c}$$
$$V_{4tt'} = Y_{tt'} \gamma_{t'} \quad \forall t, t' \in \mathcal{C}_k \tag{B.2d}$$
$$V_{5tt'} = X_t^{OC} \gamma_{t'} \quad \forall t \in \mathcal{C}_k, t' \notin \mathcal{C}_k \tag{B.2e}$$
$$V_{6tt'} = \kappa_{tt'} Z_{tt'} \quad \forall t \in \mathcal{C}_k, t' \notin \mathcal{C}_k \tag{B.2f}$$

We then substitute these variables directly into (B.1a)–(B.1e):

$$\sum_{t' \in \mathcal{C}_k} \left[\left(h_{tt'} + h_{t't} + g_{tt'} \right) \cdot V_{1tt'} \right] = \sum_{t' \in \mathcal{C}_k} \left(h_{tt'} \cdot Z_{tt'} + V_{2tt'} \right) \quad \forall t \in \mathcal{C}_k \tag{B.3a}$$

$$\sum_{t' \in \mathcal{C}_k} \left[\left(h_{tt'} + h_{t't} + g_{tt'} \right) \cdot V_{3tt'} \right] = \sum_{t' \in \mathcal{C}_k} \left(h_{tt'} \cdot \gamma_{t'} + V_{4tt'} \right) \quad \forall t \in \mathcal{C}_k \tag{B.3b}$$

$$\sum_{t' \notin \mathcal{C}_k} \left[\left(h_{tt'} + h_{t't} + g_{tt'} \right) \cdot V_{5tt'} \right] = \sum_{t' \notin \mathcal{C}_k} \left(h_{tt'} \cdot \gamma_{t'} + V_{4tt'} \right) \quad \forall t \in \mathcal{C}_k \tag{B.3c}$$

$$\sum_{t' \in \mathcal{D}_t} \left(\beta_{tt'} + V_{6tt'} \right) \le (\hat{n} - 2) + T_t^d \quad \forall t \in \mathcal{C}_k \tag{B.3d}$$

$$\sum_{t' \in \mathcal{D}_t} \left(\beta_{tt'} + V_{6tt'} \right) \ge (\hat{n} - 1) \cdot T_t^d \quad \forall t \in \mathcal{C}_k \tag{B.3e}$$

Variables $V_{1tt'}$ through $V_{6tt'}$ serve as proxy variables, which enable the linearization, and bounds are given in Constraints (B.4a)–(B.4h):

$$0 \le V_{1tt'} \le Z_{tt'} \quad \forall t, t' \in \mathcal{C}_k \tag{B.4a}$$
$$0 \le V_{2tt'} \le M \cdot Z_{tt'} \quad \forall t, t' \in \mathcal{C}_k \tag{B.4b}$$
$$0 \le V_{3tt'} \le \gamma_{t'} \quad \forall t, t' \in \mathcal{C}_k \tag{B.4c}$$
$$0 \le V_{4tt'} \le M \cdot \gamma_{t'} \quad \forall t, t' \in \mathcal{C}_k \tag{B.4d}$$
$$0 \le V_{5tt'} \le \gamma_{t'} \quad \forall t \in \mathcal{C}_k, t' \notin \mathcal{C}_k \tag{B.4e}$$
$$0 \le V_{6tt'} \le \kappa_{tt'} \quad \forall t, t' \in \mathcal{C}_k \tag{B.4f}$$
$$0 \le V_{6tt'} \le Z_{tt'} \quad \forall t, t' \in \mathcal{C}_k \tag{B.4g}$$
$$\kappa_{tt'} + Z_{tt'} - 1 \le V_{6tt'} \quad \forall t, t' \in \mathcal{C}_k \tag{B.4h}$$

We then relate variables $V_{1tt'}$ through $V_{5tt'}$ to the respective integer and continuous variables in Equations (B.5a)–(B.5e); this is an exact relationship and completes the linearization of these products of variables:

$$-(1 - Z_{tt'}) \le V_{1tt'} - X_t^T \le (1 - Z_{tt'}) \quad \forall t, t' \in \mathcal{C}_k \tag{B.5a}$$
$$-M(1 - Z_{tt'}) \le V_{2tt'} - Y_{tt'} \le M(1 - Z_{tt'}) \quad \forall t, t' \in \mathcal{C}_k \tag{B.5b}$$
$$-\left(1 - \gamma_{t'}\right) \le V_{3tt'} - X_t^{PC} \le \left(1 - \gamma_{t'}\right) \quad \forall t, t' \in \mathcal{C}_k \tag{B.5c}$$
$$-M\left(1 - \gamma_{t'}\right) \le V_{4tt'} - Y_{tt'} \le M\left(1 - \gamma_{t'}\right) \quad \forall t, t' \in \mathcal{C}_k \tag{B.5d}$$
$$-\left(1 - \gamma_{t'}\right) \le V_{5tt'} - X_t^{OC} \le \left(1 - \gamma_{t'}\right) \quad \forall t \in \mathcal{C}_k, t' \notin \mathcal{C}_k \tag{B.5e}$$

Appendix C. Defining Big *M*

The value of big M is tailored to the constraint in which it is used. For Constraint (A.2c), the value of big M must be at least 82 because the worst-case scenario is one in which one team wins no games and the other team wins all 82 games. Therefore, 82 is the smallest guaranteed win differential between two teams. Similarly, the big M in Constraint (A.3d) must also be at least 82. The big M in Constraints (A.3e) and (A.3f) and (A.4b) must be at least 82.5 to account for the extra 0.5 on the left-hand side. The big M in Constraint (A.4a) must be at least 81.5 because, when added with the 0.5 on the left-hand side, the result is at least 82. For Constraints (A.4c) and (A.4d), the big M values must be at least 7.5 and 6.5, respectively. In the former case, the largest value for the right-hand side is 7.5 because the smallest value the sum on $\beta_{t't}$ on the right-hand side can assume is 0, and $n = 8$; subtracting 0.5 and 0 from the right-hand-side value of 8 yields 7.5. In the latter case, there are 14 teams in a conference other than the team in question, so the largest value the sum on $\beta_{t't}$ on the right-hand side can assume is 14; subtracting $n = 8$ and adding 0.5 yields 6.5 as the largest right-hand-side value of this constraint. For Constraint (A.4c), the largest right-hand side occurs for the instance in which team t does not win any games, and the right-hand side is $7.5 - 0 = 7.5$. The largest right-hand side for Constraint (A.4d) occurs when team t beats the other 14 teams in the conference, and the right-hand side results in $14 - 7.5 = 6.5$. The big M is 2,352 for Constraint (A.6), because the worst case is realized when 14 teams are tied. In this scenario, the numbers of games a given team plays against the other tied teams can be different. In the worst

Table C.1. The Values for Sufficiently Large Values of Big M, and the Corresponding Rationale

Constraint number	Value of big M	Rationale
(A.2c), (A.3d)	82	Team t wins all 82 games and the other team wins no games.
(A.3e), (A.3f), (A.4b)	82.5	One team wins all 82 games and the other team wins no games plus a fraction, which we arbitrarily take to be 0.5.
(A.4a)	81.5	One team wins all 82 games and the other team wins no games minus a fraction, which we arbitrarily take to be 0.5.
(A.4c)	7.5	Team t does not beat any team in the conference.
(A.4d)	6.5	Team t beats the other 14 teams in the conference.
(A.6), (A.10)	2,352	A team plays 48 games and another plays 49 games against all tied teams.
(A.8a)	16	All teams play 16 divisional games ($\hat{d}$).
(A.9)	52	All teams play 52 conference games ($\hat{c}$).
(A.11)	30	All 15 teams from the other conference are tied for the playoffs.
(B.4b), (B.4d), (B.5b), (B.5d), (A.7d)	4	This provides the greatest number of games by which a team can beat another team.
(A.7e)	4.5	This provides the greatest number of games by which a team can beat another team plus a fraction, which we arbitrarily take to be 0.5.

case, a team plays 48 games and another plays 49 games against all tied teams. If both teams have the same number of wins, then the difference between their win percentages is $\frac{1}{48} - \frac{1}{49} = \frac{1}{2{,}352}$. For Constraint (A.6), the big M must be at least 2,352 to have the reciprocal be no greater than any difference in win percentages. In Constraint (A.8a), a team's divisional win percentage is compared with that of another tied team in the same division. All teams play 16 divisional games. Therefore, the win percentage for Constraint (A.8a) is a multiple of $\frac{1}{16}$, and with a big M of 16, $\frac{1}{16}$ is always no greater than the difference between two teams' divisional win percentages. Constraint (A.9) is similar to Constraint (A.8a), but a team's conference win percentage is compared with that of another tied team in the same conference. All teams play 52 conference games. Therefore, the win percentage for Constraint (A.9) is a multiple of $\frac{1}{52}$, and with a big M value of 52, $\frac{1}{52}$ is always no greater than the difference between two teams' conference win percentages. Constraint (A.10) has the same big M as Constraint (A.6) because 14 teams qualify or are tied for making the playoffs. Constraint (A.11) compares two teams' win percentages against those for teams in the other conference already in or tied for the playoffs. Because all teams play two games against all the teams in the other conference, the worst case for determining big M occurs when all 15 teams from the other conference are tied for the playoffs. In this case, the win percentage for Constraint (A.11) is a multiple of $\frac{1}{30}$, and with a big M value of 30, $\frac{1}{30}$ is always no greater than the difference between two teams' other-conference playoff-bound teams' win percentages. In Constraint (A.7d) and the linearization constraints, the big M is set by the greatest number of games by which a team can beat another team. No pair of teams plays more than four games against each other in a single season, so a big M value of 4 is sufficiently large. The big M in Constraint (A.7e) must be at least $4 + \epsilon$, where ϵ is a sufficiently small number. A value of 0.5 suffices for our purposes, so the sum is 4.5. The above-mentioned sufficiently large values for big M hold for regular seasons consisting of 82 games. For any season in which fewer than 82 games are played (e.g., during a season with a player lockout), these values must be reevaluated based on that season's scheduling rules. Table C.1 summarizes this discussion.

Appendix D. Mathematical Formulations for First-Place Elimination, as Well as First-Place and Playoff Clinches

Elimination from First Place

The mixed-integer program for determining whether team k has been eliminated from first place is derived from the playoff-elimination problem $(\mathcal{M}_k)$ by replacing Constraint (A.2d) with $\sum_{t \in \mathcal{C}_k} \omega_{tk} = 0$ to ensure that team k is at least tied for first place in any feasible scenario. We remove Constraints (A.13a) and (A.13b), and we add the constraint $\sum_{t \in \mathcal{C}_k} \beta_{kt} \geq |\mathcal{C}_k| - 1$. If this mixed-integer program has a feasible solution, then we cannot eliminate team k from first place, and the corresponding first-place elimination number is calculated in the same way the playoff elimination number is computed.

Clinching First Place

The mixed-integer program for determining team k's first-place clinch number is derived from problem $(\mathcal{M}_k)$ as follows:

1. The term W_k is maximized rather than minimized.
2. Constraint (A.2c) is replaced by

$$W_t + M \cdot \omega_{kt} \geq W_k + \frac{\alpha_{kt}}{2} \quad \forall t \in \mathcal{C}_k. \tag{D.1}$$

3. Constraint (A.2d) is replaced by

$$\sum_{t \in \mathscr{C}_k} \omega_{kt} \leq |\mathscr{C}_k| - 1. \tag{D.2}$$

4. Constraints (A.13a) and (A.13b) are removed.

Constraint (D.1) requires that if team t has more wins than team k, the number of wins for team t, given by W_t, must be greater than or equal to that for team k and, if they are equal, that team k wins the tie-breaker. Constraint (D.2) ensures that at least one team finishes ahead of team k in the conference standings and that if team k is tied for first place, it loses the two-way tiebreaker to the other first-place team. If the model is feasible, it maximizes the number of games team k wins without finishing in first place; if it is infeasible, it shows that team k has already clinched first place. In the former case, we report $(W_k - \hat{w}_k) + 1$ as team k's first-place clinch number.

Clinching a Playoff Spot

The playoff clinch model determines for a given team k the number of wins needed to clinch a playoff spot. We maximize the number of games team k can win, W_k, and yet fail to qualify for the playoffs. The mixed-integer program for determining team k's playoff clinch number is derived from the mixed-integer program for determining team k's first-place clinch number by replacing (D.2) with

$$\sum_{t \in \mathscr{C}_k} \omega_{kt} \leq |\mathscr{C}_k| - n. \tag{D.3}$$

Constraint (D.3) ensures that the number of teams with worse records than that of team k is at most the number of teams in the conference that fail to make the playoffs. Thus, team k misses the playoffs in the scenario either by finishing tied for the last playoff spot and losing the tiebreaker or by finishing behind at least n other teams. If the model is feasible, then RIOT reports that team k's playoff clinch number is one more than the number of games it wins in the optimal scenario, $(W_k - \hat{w}_k) + 1$. If the model is infeasible, then there is no scenario in which team k does not qualify for the playoffs; that is, team k has clinched a playoff spot.

Appendix E. The NBA's Premature Announcement of Boston's Playoff Clinch

In this section, we describe how we used the RIOT models to determine that Boston had not, in fact, clinched a playoff spot on March 8, 2018. If the playoff clinch model is feasible, then RIOT posts $(W_k - \hat{w}_k) + 1$ as the playoff clinch number. Table E.1 depicts a scenario in which Boston is in a six-way tie for five playoff spots. Note that Boston is ranked last among the tied teams because criterion 1 (winning percentage against tied teams) has the largest weight in constraint set (A.12).

Table E.1. A Scenario in Which Boston Finishes in a Six-Way Tie for Five Playoff Spots

Team	Wins	Losses	Division	Wins vs. tied teams (%)
Philadelphia	51	31	Atlantic	n/a
Toronto	51	31	Atlantic	n/a
Milwaukee	47	35	Central	n/a
Washington	46	36	Southeast	63
Cleveland	46	36	Central	61
Miami	46	36	Southeast	50
Detroit	46	36	Central	42
Indiana	46	36	Central	42
Boston	46	36	Atlantic	41

Note. n/a, not applicable.

Because the tie in Table E.1 involves more than two teams, criterion 2 (division winner) takes precedence over criterion 1. Therefore, on the basis of the logic in Figure 2, we break the tie between the two teams in the Southeast division, Washington and Miami, in Washington's favor because it won the head-to-head series with Miami three games to one in the scenario. We then recalculate the tied team winning percentages to obtain the ranking shown in Table E.2.

Table E.2. A Scenario in Which Boston Finishes in a Five-Way Tie for Four Playoff Spots

Team	Wins	Losses	Division	Wins vs. tied teams (%)
Philadelphia	51	31	Atlantic	n/a
Toronto	51	31	Atlantic	n/a
Milwaukee	47	35	Central	n/a
Washington	46	36	Southeast	n/a
Cleveland	46	36	Central	57
Miami	46	36	Southeast	57
Boston	46	36	Atlantic	46
Detroit	46	36	Central	47
Indiana	46	36	Central	44

Note. n/a, not applicable.

The logic in Figure 2 shows that Boston is actually a playoff team in the scenario. So, at that point in time, it was still an open question as to whether Boston had clinched a playoff spot. In search of a scenario in which Boston did not make the playoffs, we solve the playoff clinch model with an additional constraint to give Washington more wins than Miami. This results in the scenario described earlier and detailed in Table 3.

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Verification Letter

Alexandra M. Newman, Director, Operations Research with Engineering PhD Program, Department of Mechanical Engineering, Colorado School of Mines, Golden, Colorado 80401-1887, writes:

"Please accept this as a verification letter testifying as to the use of our models on the RIOT website at https://s2.smu.edu/~olinick/riot/basketball_main.html. Specifically, we have been tracking the number of visits to the website and have found the following:

```
Counts for the period 24/Mar/2019 through 26/Apr/2019:

Total hits for basketball_main.html     : 60
Unique IP hits for basketball_main.html : 15

Total hits for eastd.html     : 238
Unique IP hits for eastd.html : 22

Total hits for westd.html     : 149
Unique IP hits for westd.html : 23

Total hits for detail_nba_numbers.html     : 111
Unique IP hits for detail_nba_numbers.html : 22
```

"This compares somewhat favorably with the number of hits for the baseball portion of the website (despite the fact that it is yet early in the current season), whose corresponding model was published as Adler, I., Erera, A. L., Hochbaum, D. S., Olinick, E. V. (2002). 'Baseball, optimization, and the World Wide Web.' *Interfaces*, **32**(2), 12–22.

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Counts for the baseball numbers for the period 24/Mar/2019 through 26/Apr/2019:

================================================
Total hits for basketball_main.html     : 60
Unique IP hits basketball_main.html     : 15

Total hits for eastd.html               : 238
Unique IP hits eastd.html               : 22

Total hits for westd.html               : 149
Unique IP hits westd.html               : 23

Total hits for detail_nba_numbers.html  : 111
Unique IP hits detail_nba_numbers.html  : 22

Total hits for baseball_main.html       : 1903
Unique IP hits baseball_main.html       : 365

Total hits for american_league.html     : 1674
Unique IP hits american_league.html     : 242

Total hits for national_league.html     : 1874
Unique IP hits national_league.html     : 258

Total hits for detail_calc.html         : 121
Unique IP hits detail_calc.html         : 73

Total hits for faq.html                 : 67
Unique IP hits faq.html                 : 52
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"While these numbers for the NBA model, whose corresponding paper we are submitting for possible publication in *INFORMS Journal on Applied Analytics*, are relatively small, they do show interest from the general public. We expect this interest to grow in the coming seasons (the models were only posted midway through the 2018–2019 season) and with increased advertising on our part. We hope that this provides sufficient, albeit unconventional, verification regarding the use of our model."

Mark A. Husted is a professional electrical engineer at a consulting firm, Ulteig Engineering. He earned his BS in electrical engineering, MS in energy systems, and PhD in operations research with engineering at the Colorado School of Mines. Mark enjoys developing optimization models applied to energy and sports because of his passion for technology and competition.

Eli V. Olinick is an associate professor in the Department of Engineering Management, Information, and Systems at Southern Methodist University. He earned his PhD in industrial engineering and operations research at the University of California, Berkeley. A former president of the INFORMS Technical Section on Telecommunications and Network Analytics and the Dallas/Fort Worth INFORMS Chapter, his primary research interest is optimization in the context of network design for telecommunications.

Alexandra M. Newman is a professor in the Department of Mechanical Engineering, Colorado School of Mines. She specializes in deterministic optimization modeling, especially as it applies to energy and mining systems and to logistics, transportation, and routing. She received a Fulbright Fellowship to work with industrial engineers on mining problems at the University of Chile in 2010 and was awarded the 2013 INFORMS Prize for the Teaching of Operations Research and Management Science Practice.