Macroscopic Models of Traffic Flow – B

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The LWR Model

- As previously illustrated, using the conservation law, one obtains a system of three equations involving three unknown variables describing a dynamic traffic model:

$$\begin{cases} k_t + q_x = 0, \\ q = kv, \\ v = V(k), \end{cases}$$

- where $q = q(t, x)$ is flow, $k = k(t, x)$ is density, and $v = v(t, x)$ is mean traffic speed.
- If one combines the second and third equations by eliminating v, one obtains a flow-density relationship $q = Q(k)$, and the dynamic model becomes:

$$\begin{cases} k_t + q_x = 0, \\ q = Q(k), \end{cases}$$

- Or further: $k_t + Q'(k)k_x = 0$
- Where: $Q'(k) = \frac{\mathrm{d}Q(k)}{\mathrm{d}k}$
- This is the so-called LWR model to honor the three pioneers, Lighthill, Whitham, and Richards, who originally studied this problem.
- The LWR model is essentially a first-order, homogeneous, quasi-linear partial differential equation.

The LWR Model

- If we apply the results discussed previously, the LWR model with initial condition $k(0,x) = k_0(x)$ can be solved as follows:
  1. Construct a time-space diagram (i.e., the $t - x$ plane) with initial condition $k_0(x)$ labeled on the x-axis.
  2. Start with an arbitrary point on the x-axis $(0, x^*)$, and determine the k value at this point $k_0(x^*)$ and the value of $c(x^*) = Q\ (k_0(x^*))$.
  3. Draw a straight line s from point $(0, x^*)$ with slope $c(x^*)$. The line equation is $x_s = c(x^*)t + x^*$, which represents a characteristic along which the k value is constant $k(t, x_s) = k_0(x^*)$.
  4. Apply the previous two steps to other points on the x-axis and construct their corresponding characteristics.
  5. If two characteristics intersect, terminate both characteristics at their intersection and note the intersection as a point on a shock path. If a characteristic has multiple intersections, use the *Rankine-Hugonoit* jump condition to determine the right intersection. Repeat this step and find adjacent intersections. Connect these intersections to form a shock path. The solution at both sides of the shock path should be piecewise smooth with a jump along the shock path which forms a shock wave.

The LWR Model

6. If two families of characteristics diverge and, hence, leave a wedge-shaped area in between, fill this area with a fan of characteristics and construct a rarefaction wave solution in this area.
7. If an area has multiple rarefaction solutions, apply the entropy condition to select a solution that makes the most physical sense.
8. After the above steps have been followed, the solution space should be filled with characteristics. Each point in the solution space should be swept by one and only one characteristic.
9. If an arbitrary point (t, x) is of interest, one simply follows its characteristic all the way back to the x-axis and reads$k_0(x)$ off the initial condition. This$k_0(x)$ is the k value at the time-space point in question. Consequently, one finds the corresponding $q(x,t) = Q(k(t,x))$ and $v(t,x) = q(t,x)\, k(t,x)$. Hence, the solution $k(t,x)$, $q(t,x)$, and $v(t,x)$ of any time-space point (t,x) can be determined.

LWR with Greenshields Model

- The Greenshields model assumes the following linear $v - k$ relationship
  - Where v_f is free-flow speed and k_j is jam density.

$$v = v_{\mathrm{f}}(1 - \frac{k}{k_{\mathrm{j}}})$$

- This model implies the following quadratic $q - k$ relationship

$$q = Q(k) = v_{\mathrm{f}}\left(k - \frac{k^2}{k_{\mathrm{j}}}\right) \quad \Rightarrow \quad c(k) = Q'(k) = v_{\mathrm{f}} - 2\frac{v_{\mathrm{f}}}{k_{\mathrm{j}}}k$$

- If the parameters are traffic speed $v_f = 60$ miles per hour and density $k_j = 240$ vehicles per mile, the explicit form of the LWR model becomes:

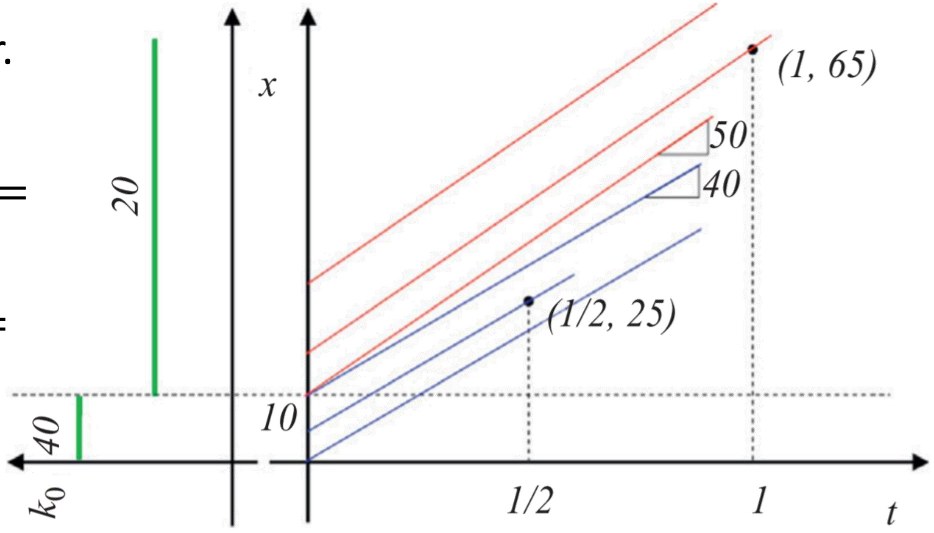
$$k_t + (60 - \frac{k}{2})k_x = 0$$

- Find solutions at points ($t = 1/2h, x = 25miles$) and ($t = 1h, x = 65miles$) with use of the following initial condition:

$$k(0, x) = k_0(x) \begin{cases} 40 \text{ vehicles per mile} & \text{if } 0 < x \leq 10 \text{ miles,} \\ 20 \text{ vehicles per mile} & \text{if } x > 10 \text{ miles.} \end{cases}$$

LWR with Greenshields Model

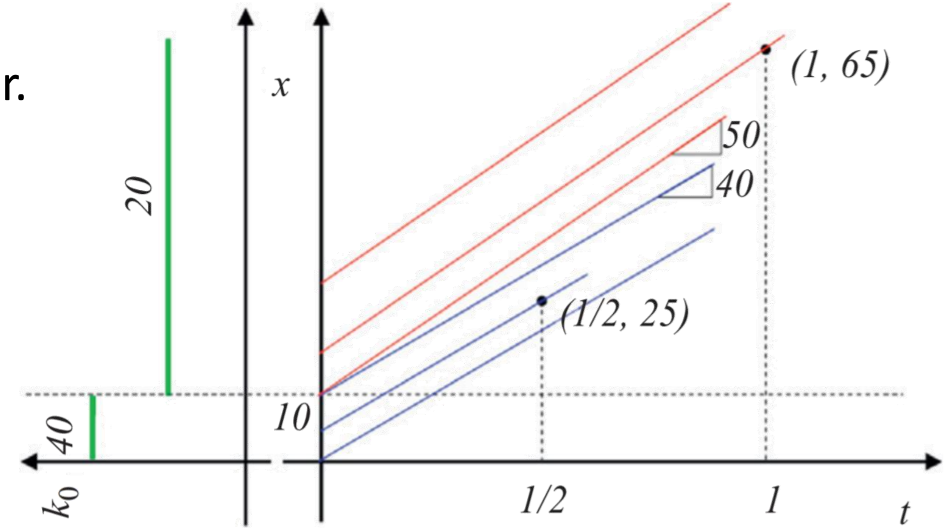
- Construct a time-space diagram.
- Illustrate the initial condition at the side of the diagram.
- Identifies the two points in question.
- Constructs characteristics.
  - All characteristics drawn between $0 < x \leq 10$ miles will bear a k value of 40 vehicles per mile, which can be read from the initial condition, so the slope of these characteristics is $c = 60 - \frac{k}{2} = 40$ miles per hour.
  - Point $(t = 1/2, x = 25)$ is within this area, and the characteristic passing this point intercepts the x-axis at $(0, 5)$. Hence, $k(1/2, 25) = k(0, 5) = 40$ vehicles per mile.
  - Similarly, All characteristics drawn from $x > 10$ miles have slope $c = 50$ miles per hour, and point $(t = 1, x = 65)$ is within this area.
  - The characteristic passing this point intercepts the x-axis at $(0, 15)$. Hence, $k(1, 65) = k(0, 15) = 20$ vehicles per mile.



Example of LWR with Greenshields model

LWR with Greenshields Model

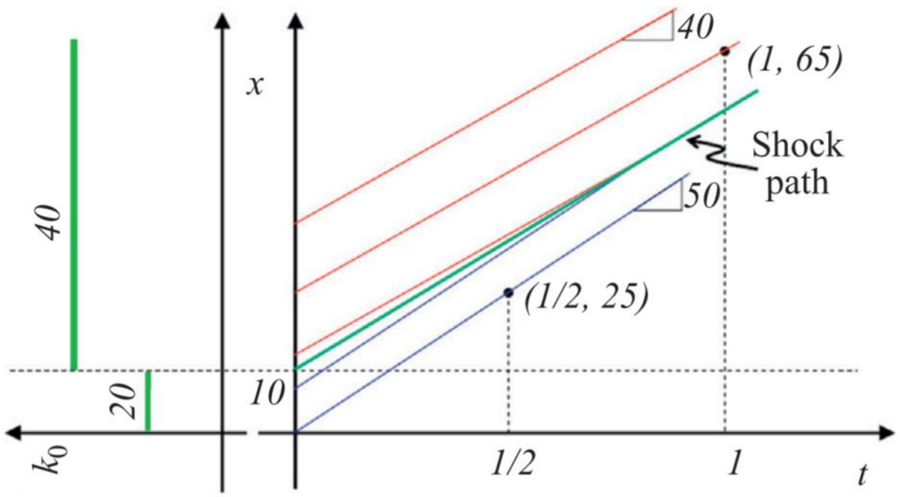
- This example involves two platoons: A fast one running in front and a slow one trailing behind.
- Each platoon corresponds to a family of characteristics called a kinematic wave.
- The characteristics of the fast platoon have a slope of 50 miles per hour, which is the speed of the fast kinematic wave.
- Similarly, the speed of the slow kinematic wave is 40 miles per hour.
- Noticeably, there is a wedge between the two families of characteristics starting from (0, 10), meaning there is an increasing "vacuum" (or gap) between the two platoons.



Example of LWR with Greenshields model

LWR with Greenshields Model

- If the two platoons are reversed—that is, the slow platoon leads the fast platoon, sooner or later the fast platoon will catch up with the slow platoon.
- When this occurs, the first vehicle in the fast platoon will have to adopt the speed of the last vehicle in the slow platoon.
- Shortly afterward, the second vehicle in the fast platoon will have to slow down, and so will the third vehicle, the fourth vehicle, and so on.
- The "slowing down" effect will propagate backward along the fast platoon.
- The propagation of a sudden change of traffic condition creates a shock wave which delineates regions of different traffic conditions.
- The trajectory of the shock wave in the $x-t$ plane is called a shock path.



Example of LWR with Greenshields model

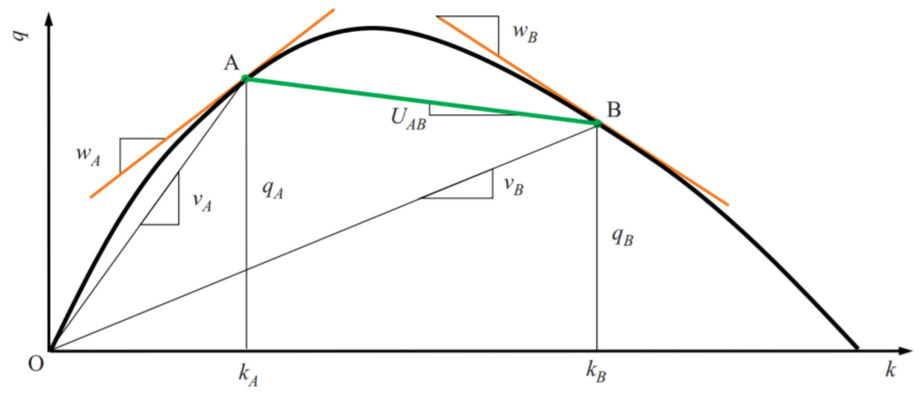
LWR Model – General Q-K Relationship

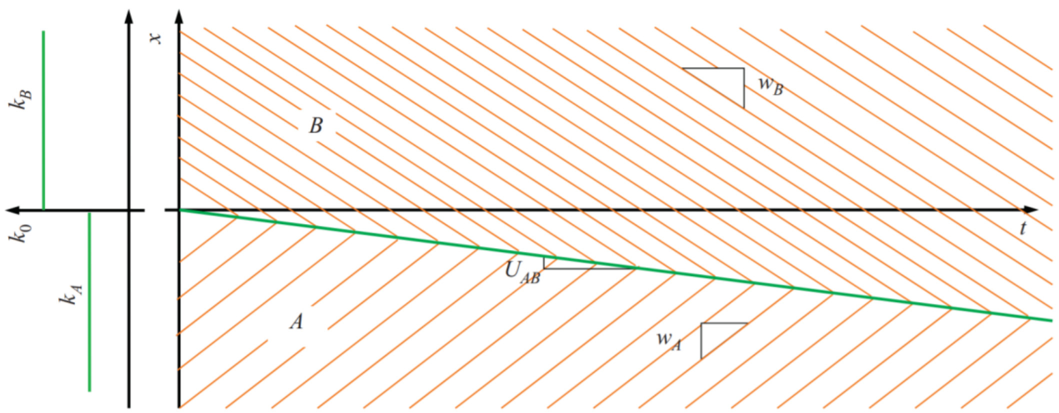
- In the above example, the underlying $q - k$ relationship is explicitly given by the Greenshields model.
- Hence, it is convenient to determine the speed of a kinematic wave from the initial condition.
- However, it is recognized that the Greenshields model suffers from inaccuracy, and often the underlying $q - k$ relationship is graphically given by fitting from empirical data.
- In this case, the solution to the LWR model with a general $q - k$ relationship is typically determined graphically.
- Consider the following LWR model with a general $q - k$ relationship:

$$\begin{cases} k_t + q_x = 0, \\ q = Q(k), \\ k(t,0) = k_0(x) = \begin{cases} A & \text{if } x \leq 0, \\ B & \text{if } x > 0, \end{cases} \end{cases}$$

LWR Model – General Q-K Relationship

- The $q - k$ relationship is illustrated in Figure.
- Point A denotes an operating point characterized by flow q_A, density k_A, and speed v_A, and similar notation applies to point B.
- A time-space diagram is constructed below the $q - k$ relationship with the initial condition at the side.
- As discussed Previously, if c is a constant or dependent on k but not explicitly dependent on t or x, the resultant characteristic is a straight line.
- Each kinematic wave has a constant slope, and the shock path will be a straight line.
- From the initial condition, there are two kinematic waves: kinematic wave A emitted from $x \leq 0$, and kinematic wave B emitted from $x > 0$.





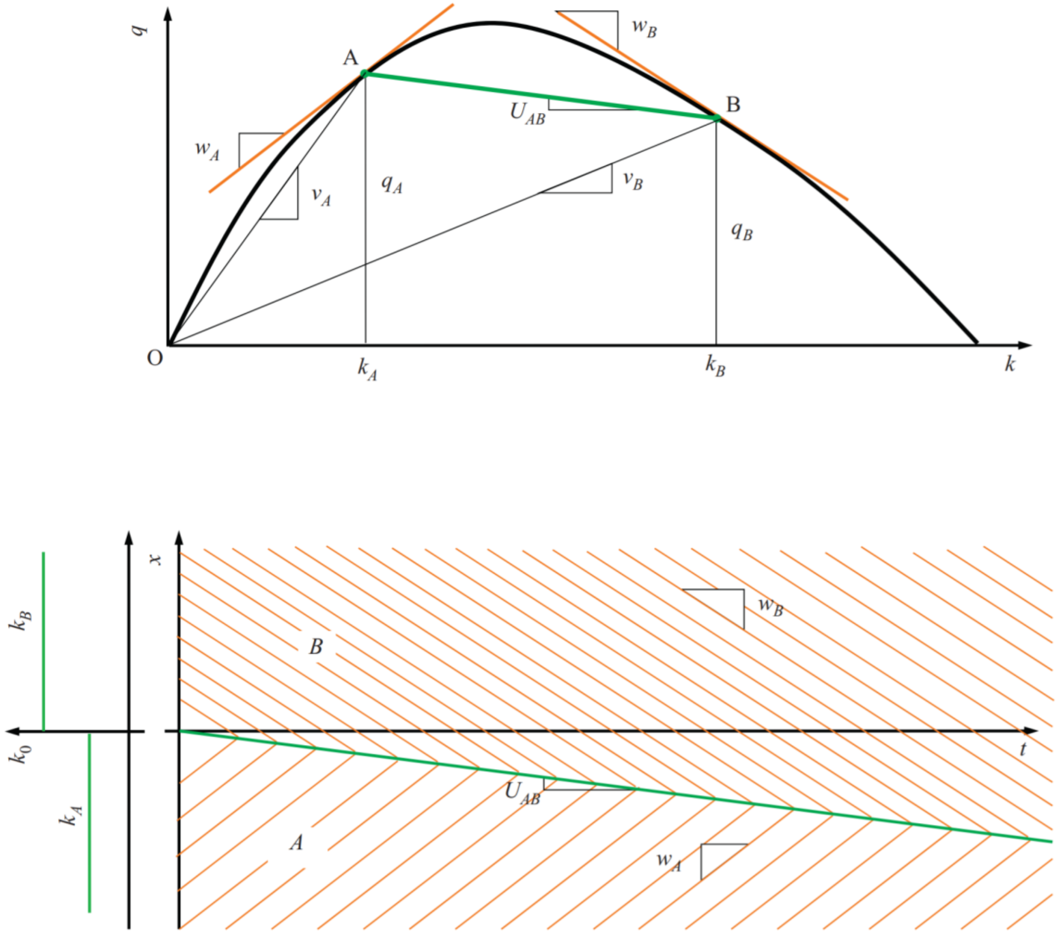
Example of LWR model with a general q-k relationship

LWR Model – General Q-K Relationship

- The speed of kinematic wave A is the derivative of the $q-k$ relationship evaluated at operating point A.

$$w_A = \frac{\mathrm{d}Q}{\mathrm{d}k} = Q'(k)|_{k=k_A}$$

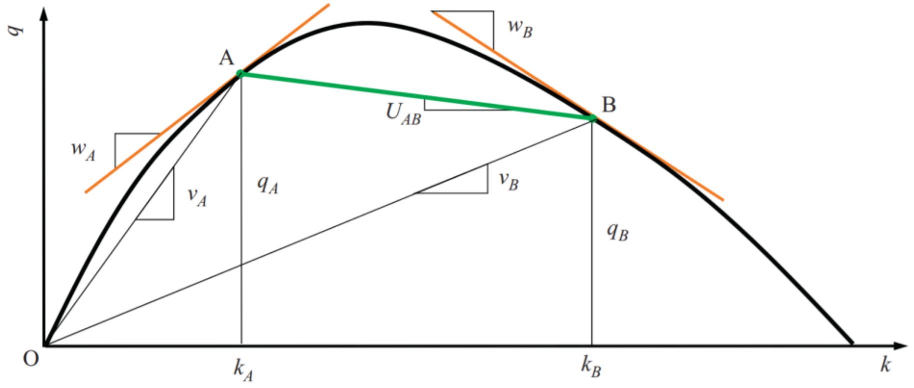
- This is the tangent to the $q-k$ curve at point A.
- Therefore, one constructs kinematic wave A by drawing a family of straight, parallel lines drawn from $x \leq 0$ with slope w_A.
- Similarly, the speed of kinematic wave B, w_B, is the tangent to the $q-k$ curve at point B, and the wave can be constructed accordingly.

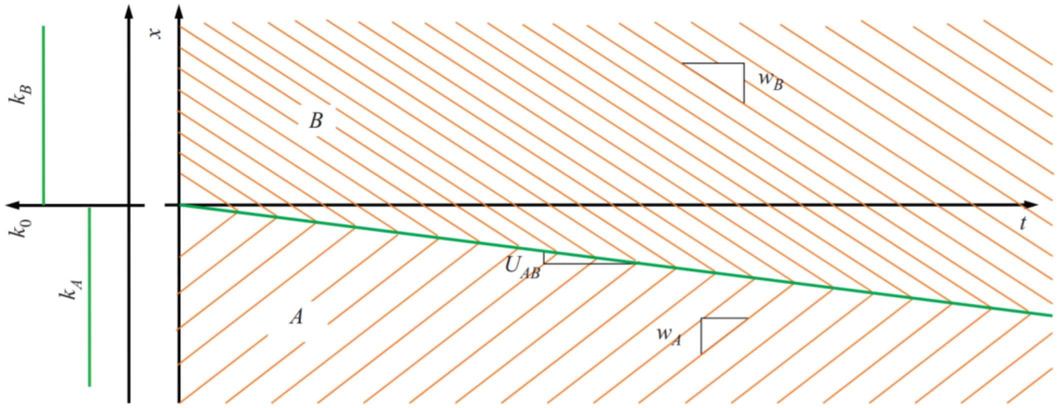


Example of LWR model with a general q-k relationship

LWR Model – General Q-K Relationship

- Kinematic wave B represents a heavy, slow platoon in front.
- Kinematic wave A represents a light, fast platoon behind.
- Kinematic wave A will catch up with kinematic wave B, creating a shock wave.
- Since the slope (speed) of the shock wave is determined by the *Rankine Hugonoit* jump condition, the shock path is a straight line.
- The slope of this line is determined using: $U_{AB} = \frac{q_B - q_A}{k_B - k_A}$
- This happens to be the slope of the line connecting points A and B in the $q - k$ curve.
- In addition, one already knows from the initial condition that the shock path starts at the origin in the time-space diagram. Therefore, one can determine the shock path by drawing a line from the origin with slope U_{AB}.

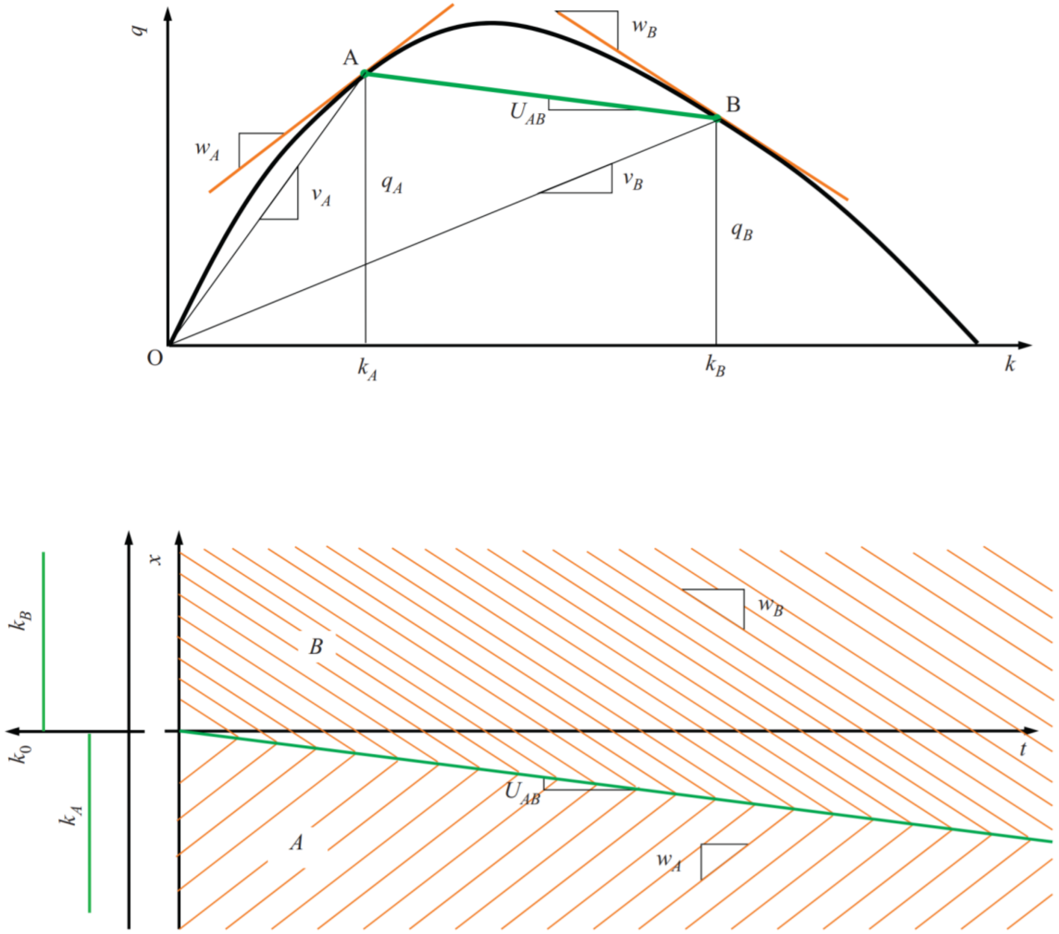




Example of LWR model with a general q-k relationship

LWR Model – General Q-K Relationship

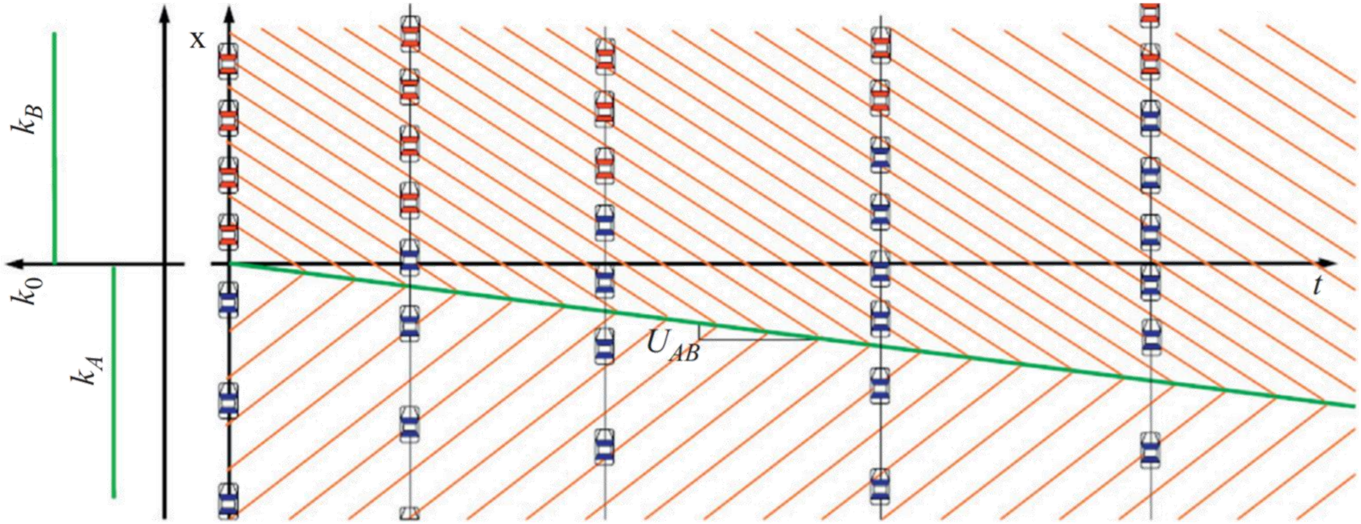
- Characteristics in the two kinematic waves will terminate once they meet the shock path.
- Hence, the shock wave solution is graphically constructed, and consists of two piecewise smooth solutions.
- The region above the shock path has a uniform traffic condition $B\ (q_B, k_B, u_B)$.
- The region below the shock path has condition $A\ (q_A, k_A, u_A)$.
- Although characteristics are used to illustrate how to find the shock path, they are not necessary.
  - With a known point on the shock path and known shock speed, the shock path can be determined directly without characteristics being drawn.



Example of LWR model with a general q-k relationship

Shock Path and Queue Tail

- As illustrated in the Figure, the shock path represents the time-varying location, which separates the fast platoon and the slow platoon ➔ The tail of a moving queue.
- As the leading vehicle of the fast platoon catches up with the tail of the slow platoon, that vehicle joins the slow platoon and becomes its new tail.
- Since the slow platoon is still moving, the location of its tail changes dynamically depending on how quick the fast platoon arrives.
- Figure shows a few snapshots to illustrate such a dynamic process.

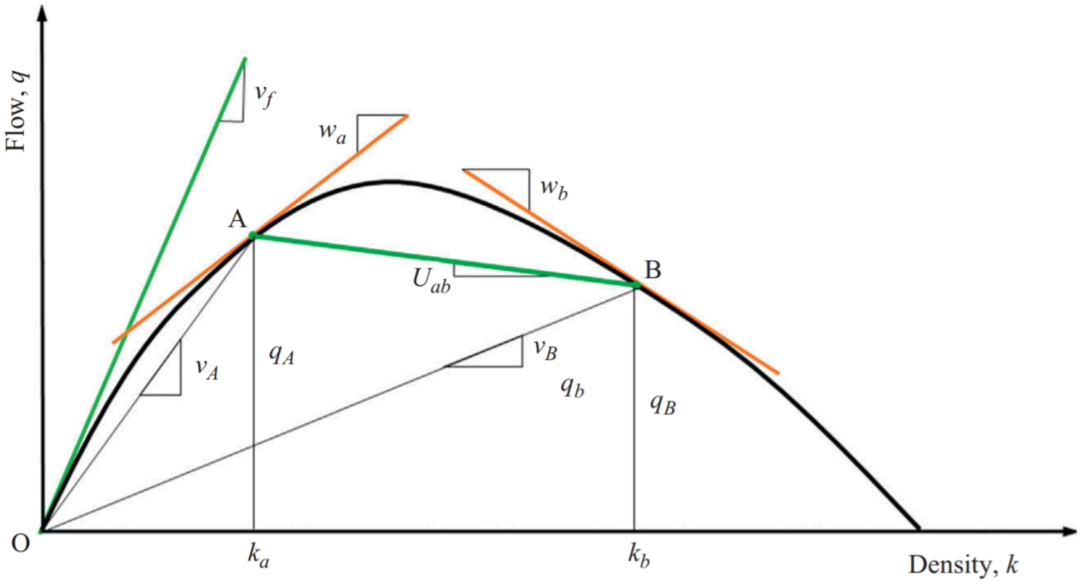


Shock path and queue tail

Speed-Flow-Density Relationship

- Given a stationary observer of the traffic flow:
  - Point A represents the traffic condition with flow q_A and density k_A
  - The corresponding traffic speed under condition A, by definition:
    $$v_A = \frac{q_A}{k_A}$$
  - Graphically, this can be represented as the slope of the line connecting the origin O and operating point A.
  - If k_A decreases, point A will move along the curve toward the origin O. In the limiting case where $k_A \rightarrow 0$, line OA becomes the tangent to the curve at the origin. The slope of this tangent denotes the traffic speed when the density is close to zero.

$$v_f = \lim_{A \rightarrow O} v_A = \lim_{k_A \rightarrow 0} \frac{q_A}{k_A}$$



Speed-flow-density graphical relationship

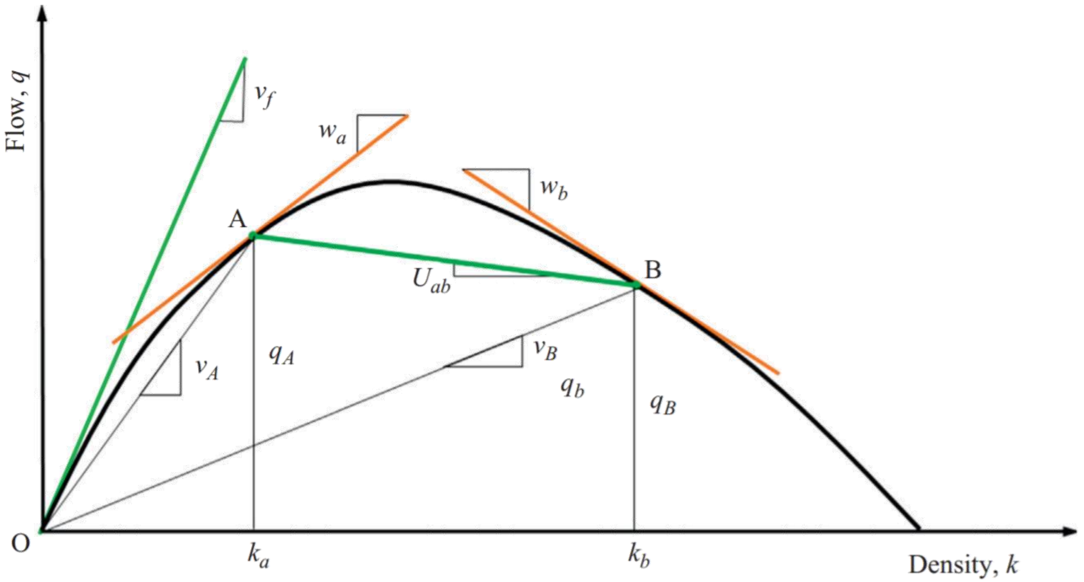
Speed-Flow-Density Relationship

- If one draws a line tangent to the curve at point A, as discussed above, the slope of this tangent is the speed of a kinematic wave carrying traffic condition A:

$$w_A = Q'(k)|_{k=k_A}$$

- If A and B represent two different traffic conditions, as discussed above, the slope of chord AB is the speed of the shock wave separating the two traffic conditions.

$$U_{AB} = \frac{q_B - q_A}{k_B - k_A}$$



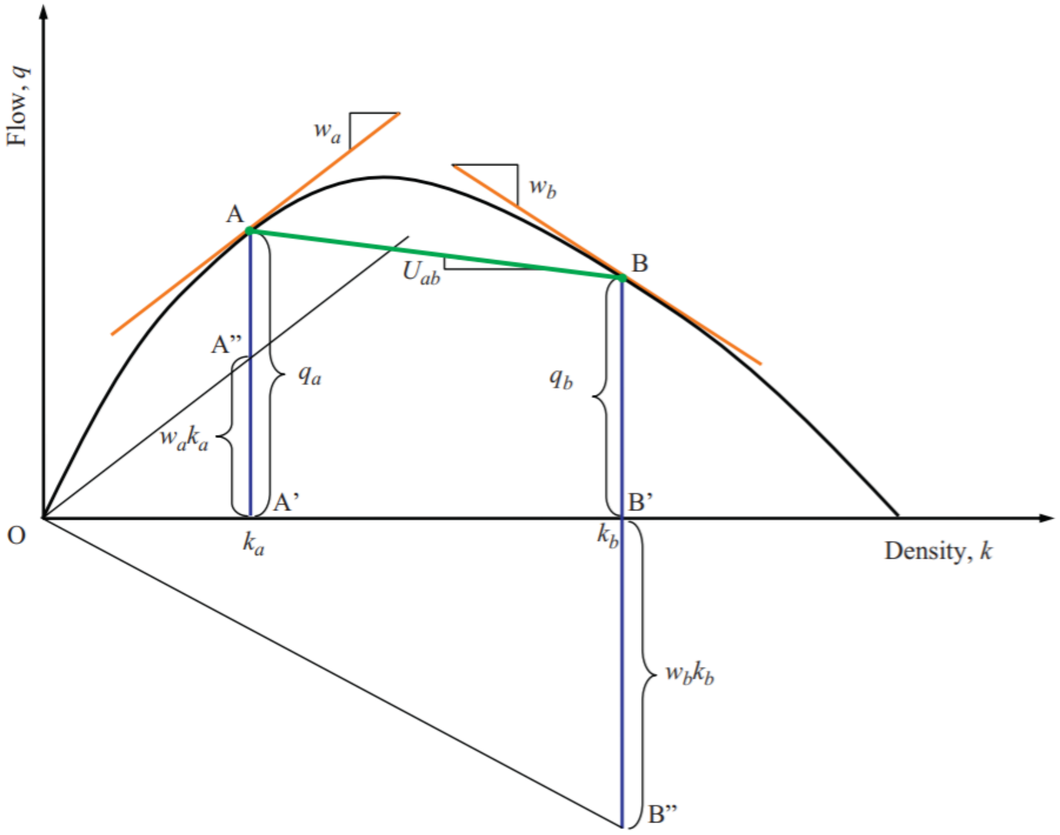
Speed-flow-density graphical relationship

Speed-Flow-Density Relationship

- Given a moving observer of the traffic flow:
  - If the observer is riding on the kinematic wave carrying traffic condition A, he will observe less flow than he would have observed if stationary:

$$\tilde{q}_A = q_A - w_A k_A$$

  - This is equivalent to drawing a line from the origin with w_A as its slope.
  - Then run a vertical line through point A intersecting the drawn line at A' and the horizontal axis at A''.
  - The length of AA' is q_A, the segment of $A'A''$ is $w_A k_A$, and the segment of AA'' is the relative flow, $\tilde{q}_A$, observed by the moving observer.



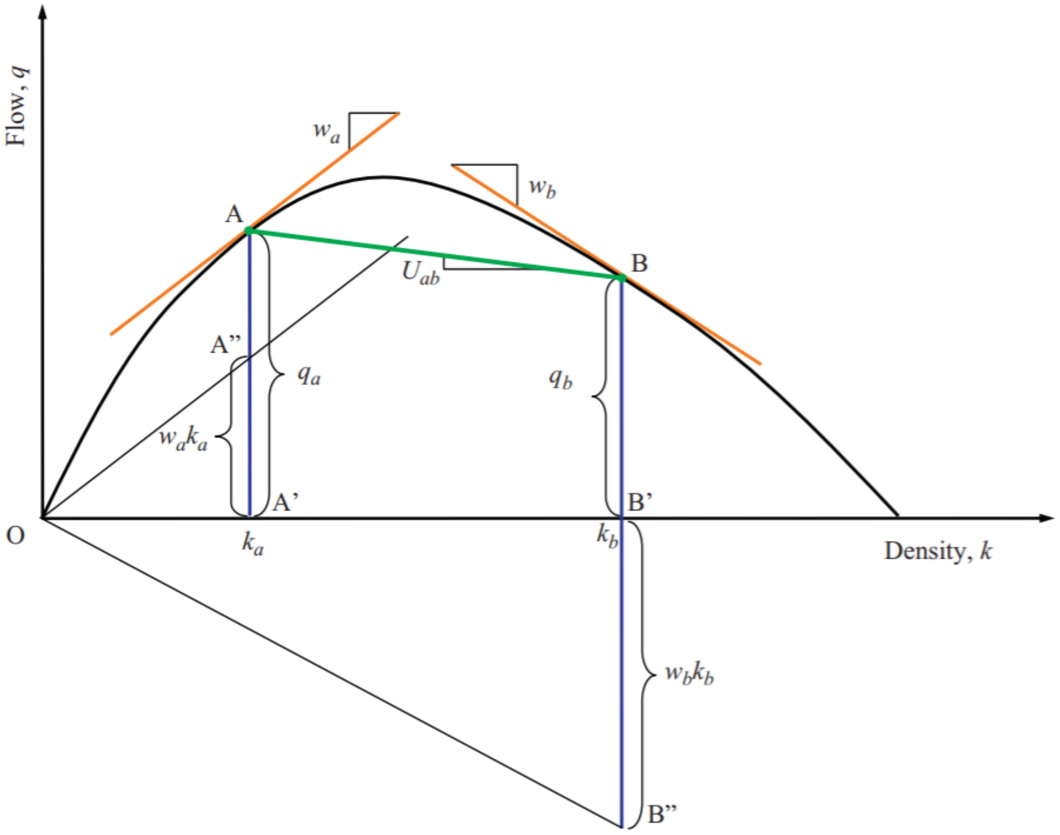
Speed-flow-density graphical relationship

Speed-Flow-Density Relationship

- As another example, suppose traffic is operating at condition B which is on the congested side of the $q-k$ curve. The kinematic wave speed is now w_B, which is negative. What happens if an observer is moving along with wave w_B?
- With the same treatment, one obtains

$$\tilde{q}_B = q_B - w_B k_B$$

- This is equivalent to drawing a line from the origin O with slope wB which slants downward.
- Run a vertical line through point B intersecting the drawn line at B'' and the horizontal axis at B'.
- The absolute value of relative flow (i.e., the length of BB'') in this case is the sum of $BB'$and $B'B''$ because w_B takes a negative value.



Speed-flow-density graphical relationship

LWR Model – Bottleneck

Problem

- Traffic arriving at the upstream point of a highway was initially under condition A (see Table).
- At $9{:}00\ a.m.$, the arriving traffic switches to condition B.
- After $1\ h$, the arriving traffic switches back to condition A.
- The capacity at the bottleneck is 1400 vehicles per hour.

Find how far the queue extends back and how long the queue persists.

Condition	*q* (vehicles/h)	*k* (vehicles/km)	*v* (km/h)
A	600	8.57	70
B	2000	40	50
D	1400	21.5	65
D’	1400	130	10.8

LWR Model – Bottleneck

Solution

- The rate at which the queue grows is:

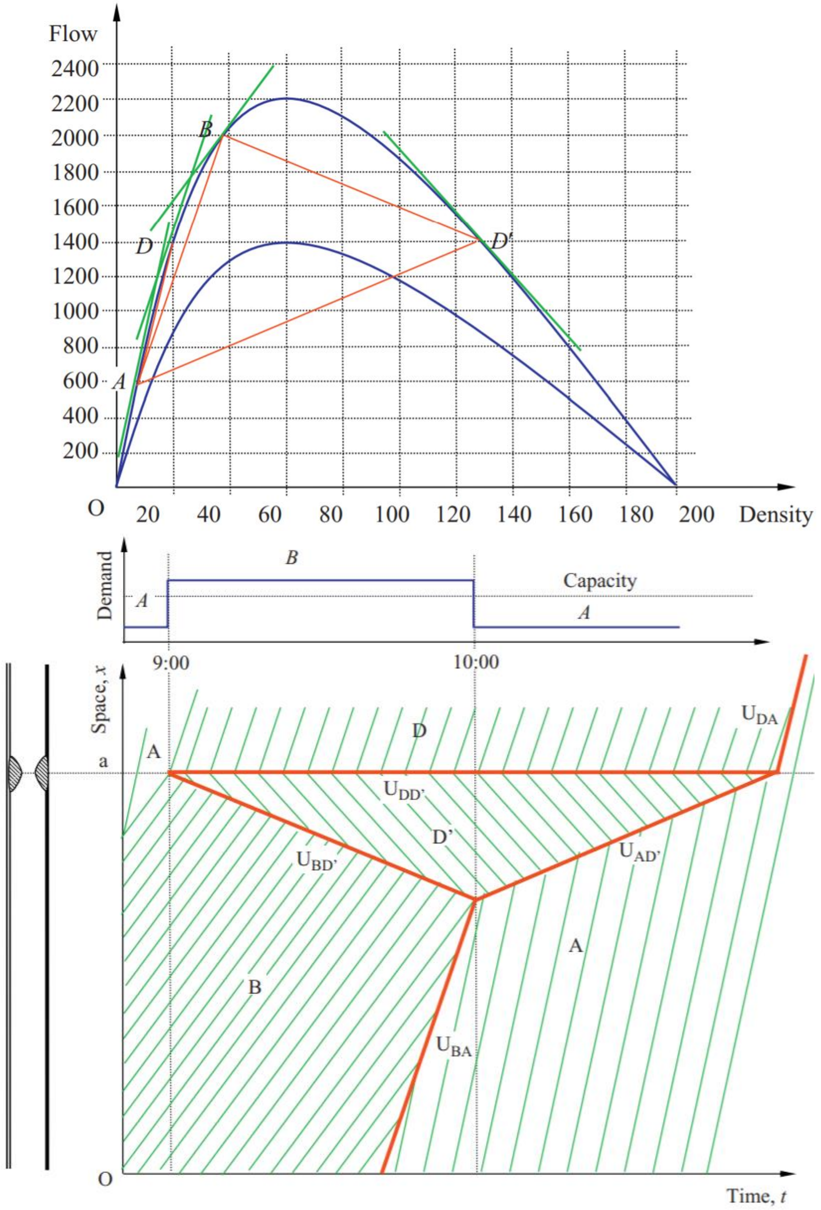
$$U_{BD'} = \frac{q_{D'} - q_B}{k_{D'} - k_B} = \frac{1400 - 2000}{130 - 40} = -\frac{600}{90} = -6.67\,\mathrm{km/h}$$

The queue tail extends back at this rate for 1 h, so the farthest point it reaches is 6.67 km upstream of the bottleneck.

- The rate at which the queue dissipates is:

$$U_{AD'} = \frac{q_{D'} - q_A}{k_{D'} - k_A} = \frac{1400 - 600}{130 - 8.57} = 6.60\,\mathrm{km/h}$$

- So the time needed to dissipate the queue is $6.67/6.60\ =\ 1.01\ h$, and the total time for which the queue persists is $2.01\ h$.



A highway bottleneck with varying traffic demand

LWR Model – Moving Bottleneck

Problem

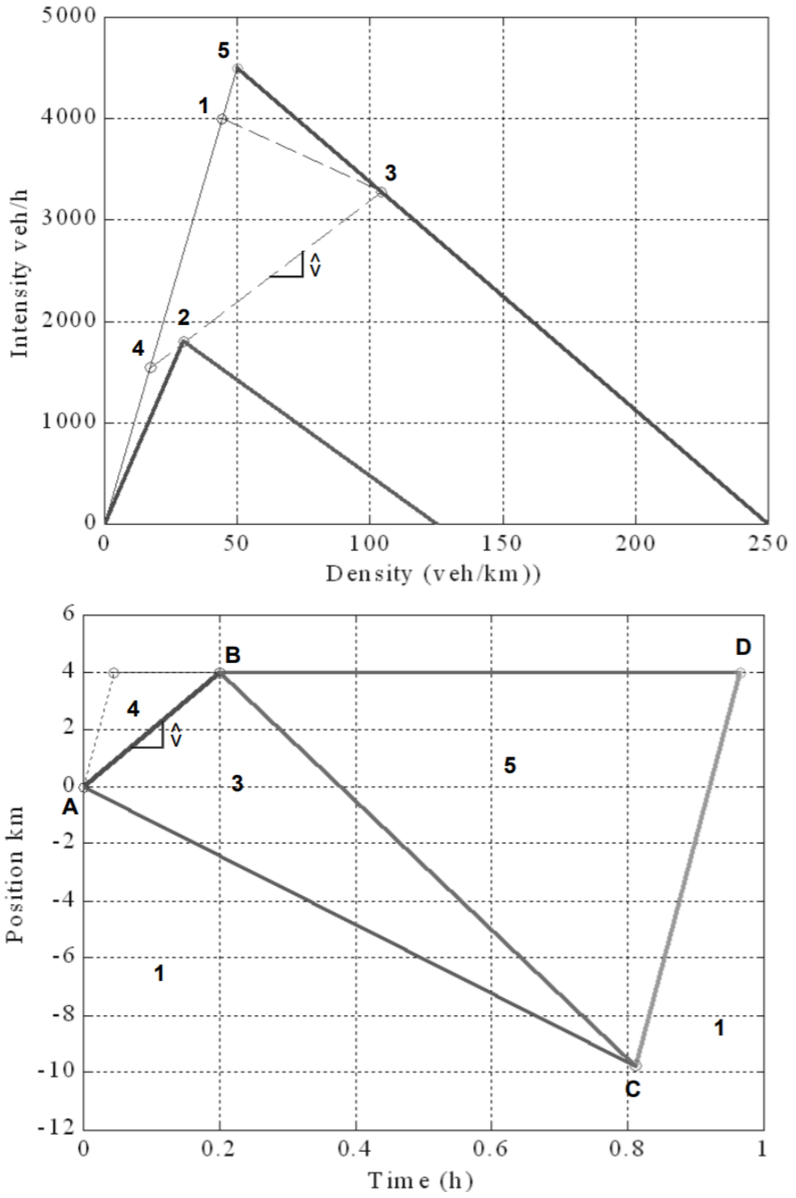
- In this section we consider a more complicated example, namely a moving bottleneck that is present on a road over a given distance during a given period.
- A practical example of a moving bottle-neck is a slow-moving vehicle, e.g. an agricultural tractor with a speed of 20 km/h, on a two-lane road. The capacity of the bottleneck is determined by the overtaking opportunities, hence by the opposing flow and the overtaking sight distance. Those two factors can lead to a more or less constant capacity of the b-n.
- Consider the following conditions:
- Roadway:
  - Capacity = 4500 veh/h;
  - Speed u_c = 90 km/h;
  - Jam density k_j = 250 veh/km.
- The moving bottleneck
  - Speed $\hat{v}$ = 20 km/h
  - Distance = 4 km.
  - Capacity = 1800 veh/h,
  - Capacity speed = 60 km/h
  - k_j = 125 veh/km

Find when the impact of the truck will disappear.

LWR Model – Bottleneck

Solution

- In general terms the bottleneck will lead to congestion upstream and free flow downstream.
- The downstream flow is expected to be equal to the capacity of the bottleneck.
- After the bottleneck is removed the congestion will decay until free flow is restored over the total affected road section.
- It is obvious that in the bottleneck the traffic flow state is the cap: state 2.
- At the upstream end of the bottleneck a transition from a congested state 3 to the state 4 must occur and move with speed $\hat{v}$.
- This implies that in the $q - k$ plane the shock waves are represented by the line going through the capacity point of the bottleneck with a slope equal to $\hat{v}$.
- Consequently the state in the congestion upstream the bottleneck is state 3 and the state downstream the bottleneck is state 4.

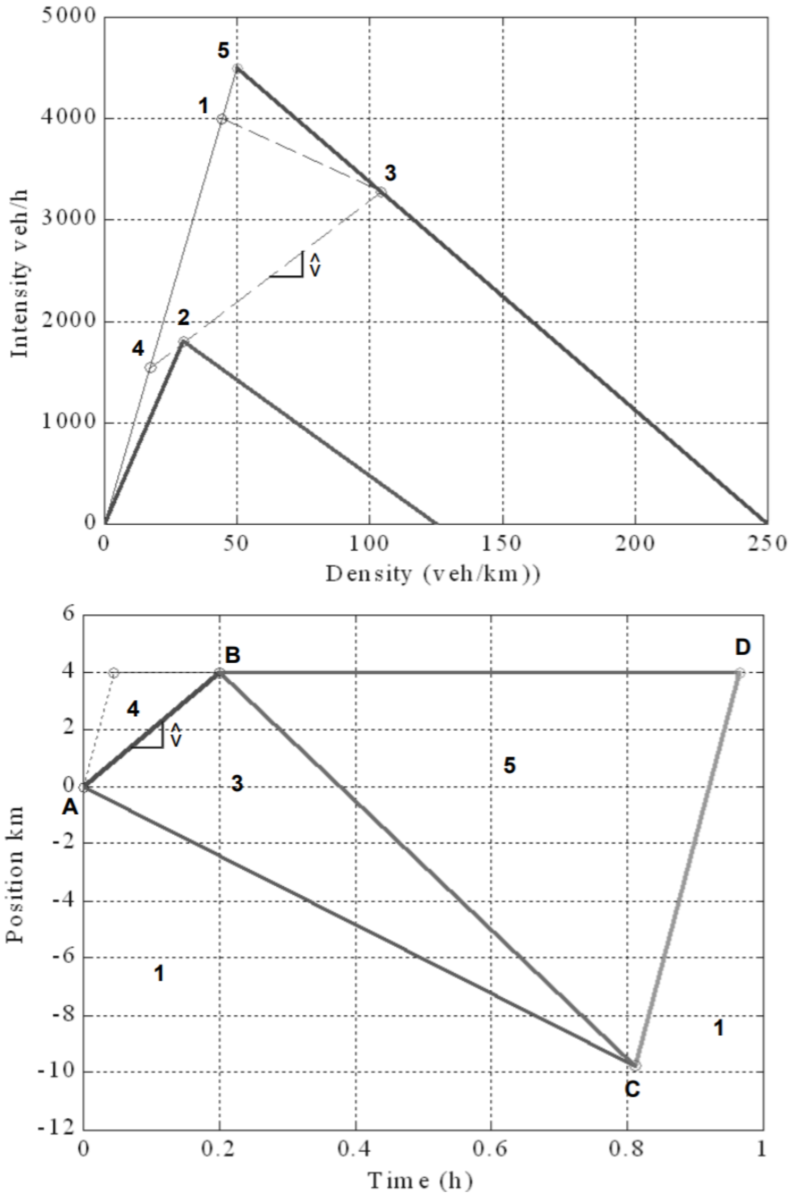


A moving bottleneck with constant demand

LWR Model – Bottleneck

Solution

- Note that the flow of the free flow downstream the bottleneck is not equal to the capacity of the bottleneck, but it is less.
- Free flow state 1 and congested state 3 determine the speed of the shock wave at the tail of the queue (a shock wave with speed ω_{13}).
- The origin in the x-t diagram is where the bottleneck starts.
- It moves with speed $\hat{v}$ until it leaves the road again.
- At a given moment and position the moving bottleneck leaves the road (point B).
- The congested traffic, state 3, then transforms to the capacity state 5 and a shock wave between state 3 and state 5 starts and makes the size of congestion shrink.

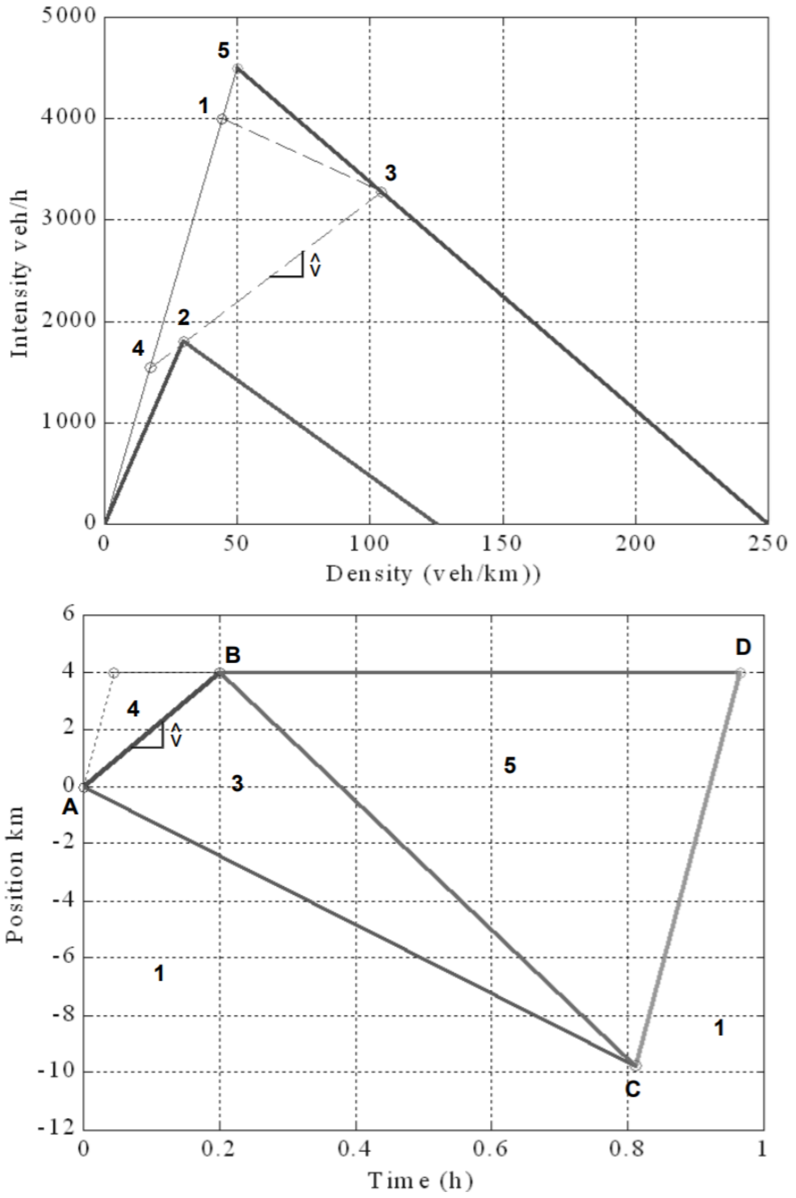


A moving bottleneck with constant demand

LWR Model – Bottleneck

Solution

- Shock waves S_{13} and S_{53} meet at point C, and then starts a shock wave representing the transition from the undisturbed state 1 to capacity state 5.
- Line AB represents a shockwave with the speed ω_{24}.
- Note: downstream point B, Daganzo's diagram works out less realistically. The three states 4, 5, and 1 propagate all with the same speed and do not mix. In reality, $u_4 > u_1 > u_5$ and drivers will accelerate to areas with a lower density.



A moving bottleneck with constant demand

Types of Shock Waves

- Several types of shock waves can be formed, depending on the traffic conditions that lead to their formation.

Frontal stationary shock waves

- Formed when the capacity suddenly reduces to zero at an approach or set of lanes, for example the red indication at a signalized intersection or when a highway is completely closed because of a serious incident.

Backward forming shock waves

- Formed when the capacity is reduced below the demand flow rate resulting in the formation of a queue upstream of the bottleneck. This may occur on a highway where the number of lanes is reduced.

Backward recovery shock waves

- Formed when the demand flow rate becomes less than the capacity of the bottleneck or the restriction causing the capacity reduction at the bottleneck is removed. The intersection of the backward forming shock wave and the backward recovery shock wave indicates the end of the queue.

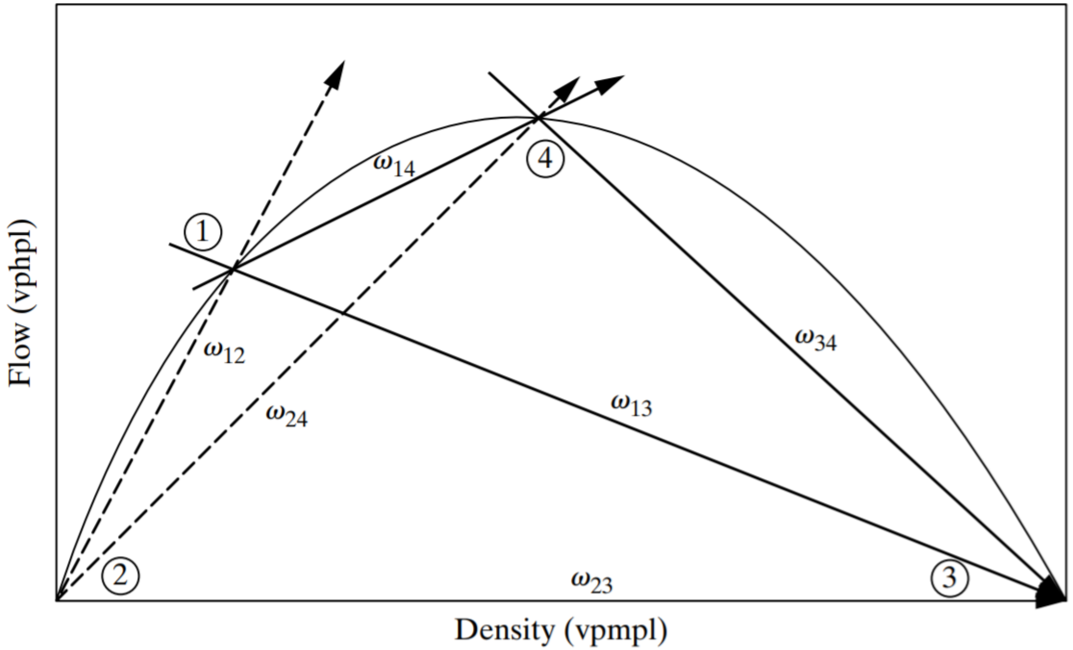
Rear stationary and forward recovery shock waves

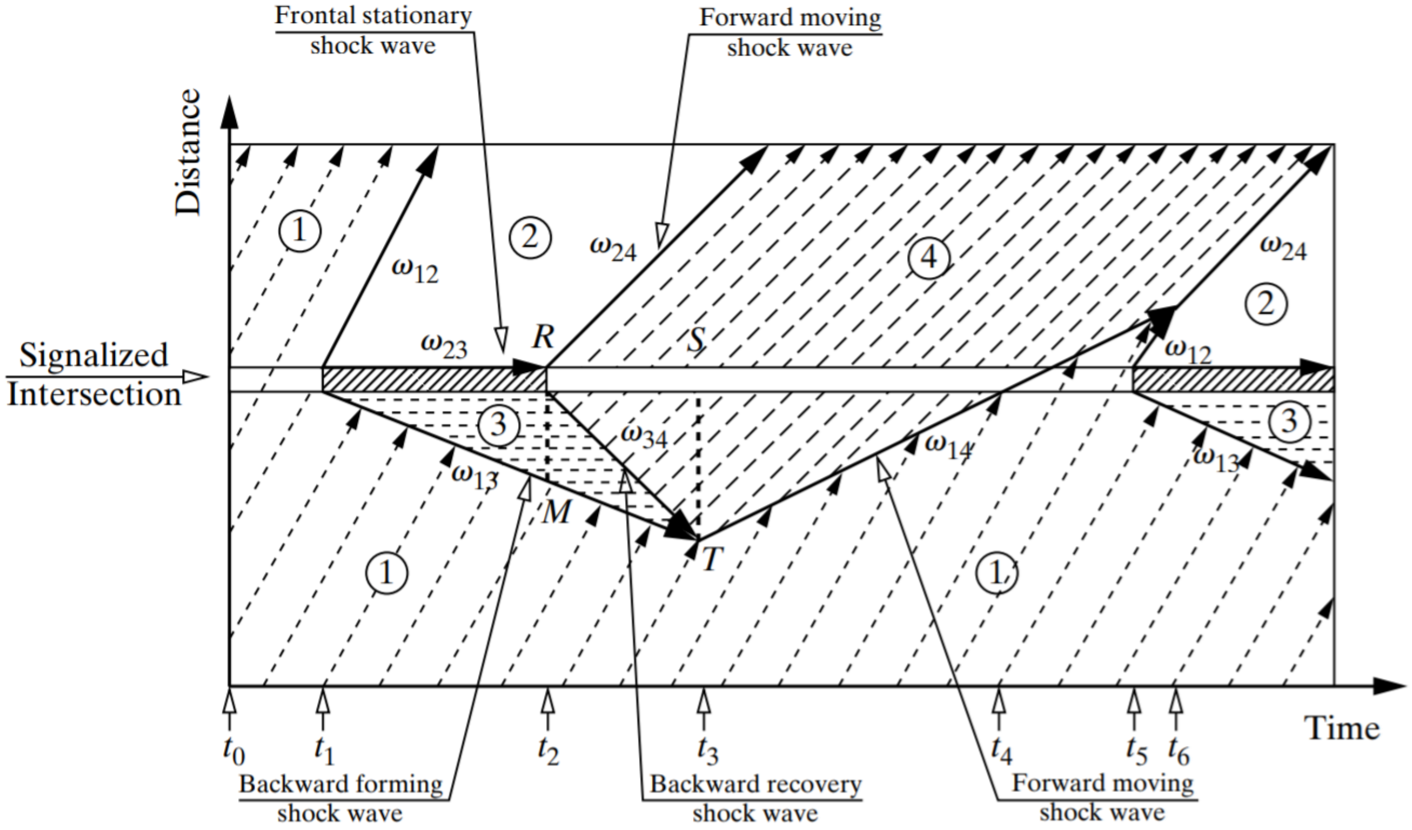
- formed when demand flow rate upstream of a bottleneck is first higher than the capacity of the bottleneck and then the demand flow rate reduces to the capacity of the bottleneck

Types of Shock Waves

Types of Shock Waves at Signalized Intersection

The red signal reduced the capacity to zero resulting in the formation of a frontal stationary shock wave.





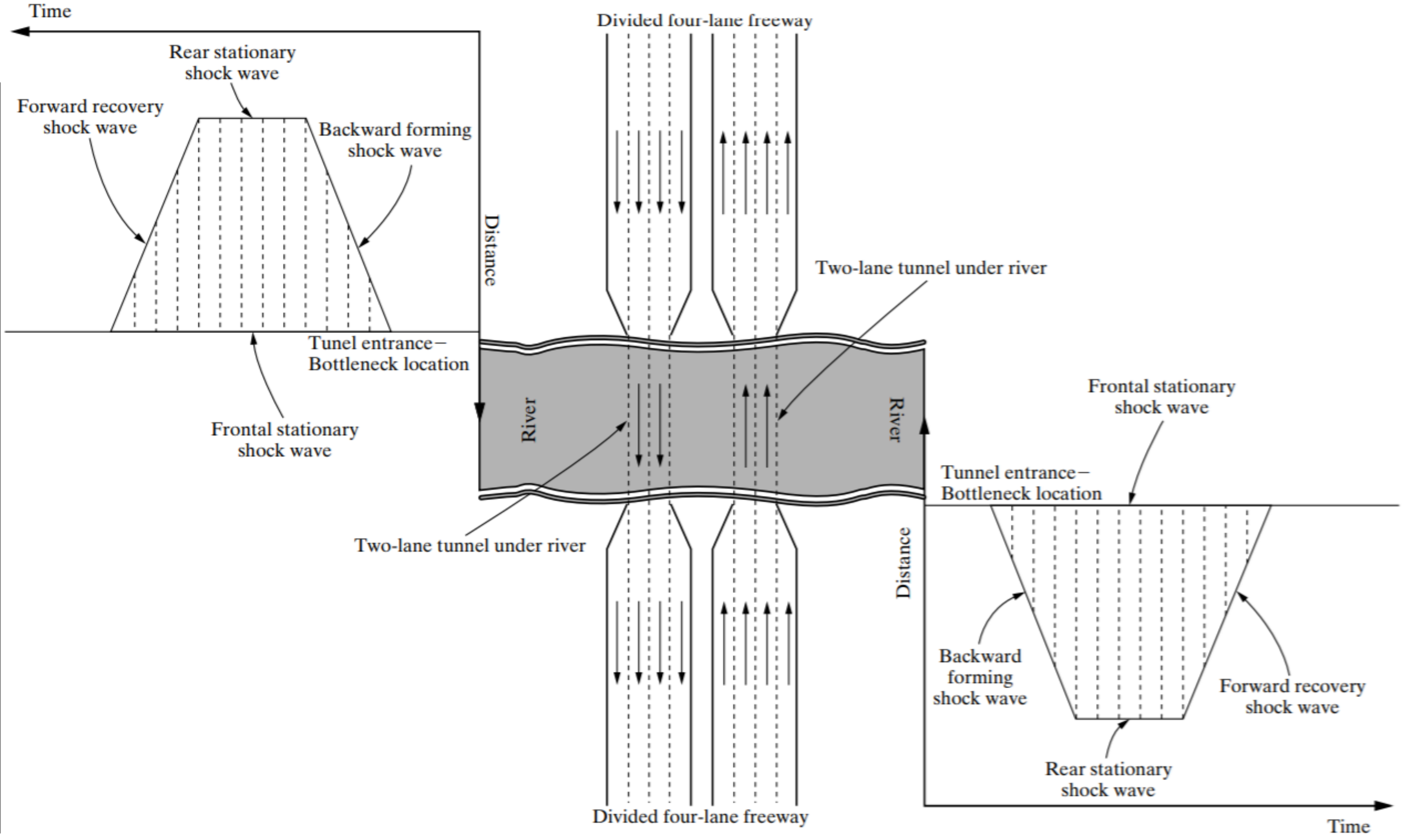
The capacity reduction below the demand flow rate results in the formation of a queue upstream of the bottleneck.

The signals turns green, the traffic on that approach is free to move across the intersection.

Types of Shock Waves

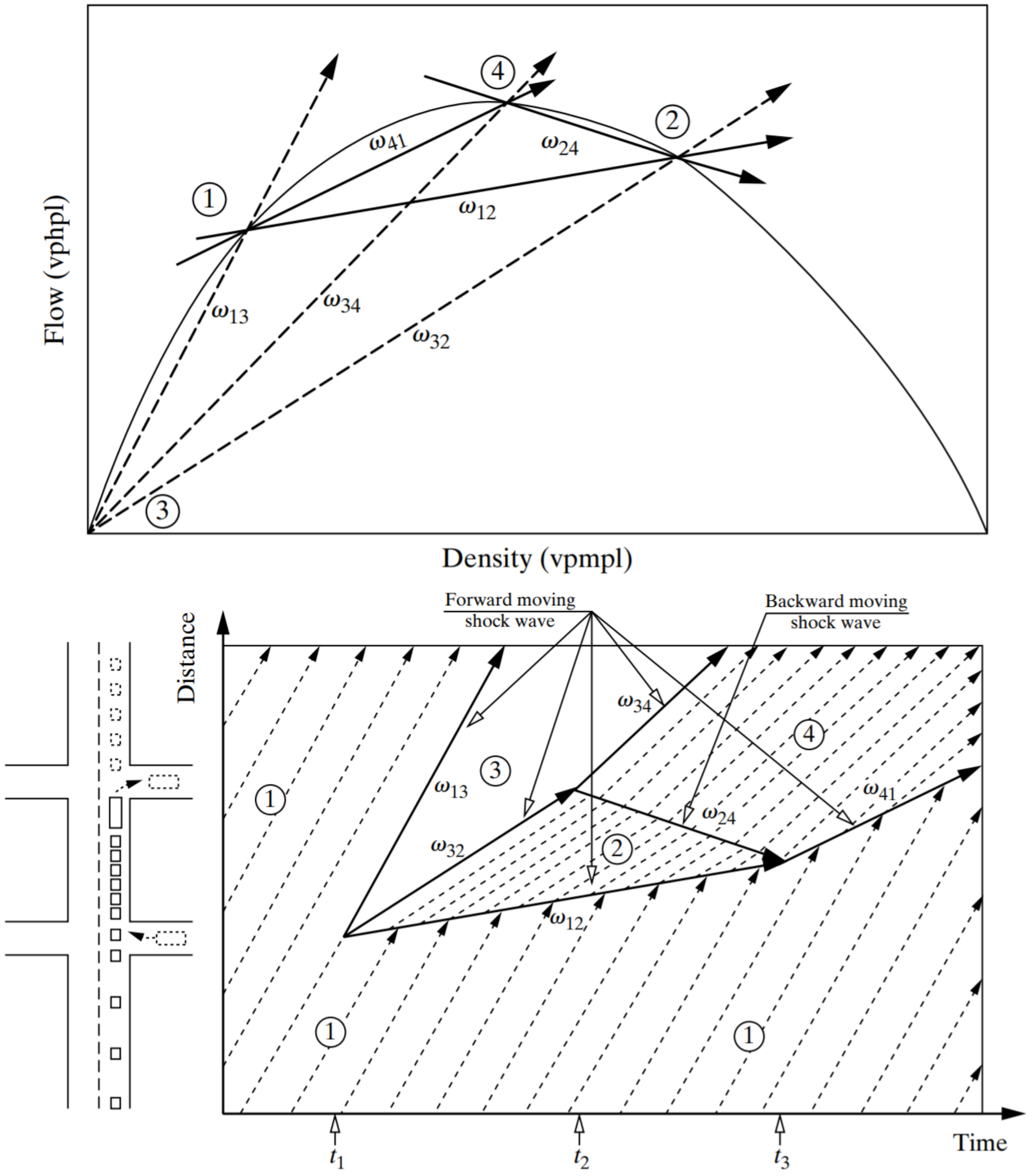
Types of Shock Waves Due to a Bottleneck

- During the off-peak period when the demand flow is less than the tunnel capacity, no shock wave is formed.
- When demand becomes higher than the tunnel capacity during the peak hour, a backward forming shock wave is formed.
- This shock wave continues to move upstream of the bottleneck as long as the demand flow is higher than the tunnel capacity.
- As the end of the peak period approaches, the demand flow rate tends to decrease until it is the same as the tunnel capacity.
- At this point, a rear stationary shock wave is formed until the demand flow becomes less than the tunnel capacity resulting in the formation of a forward recovery shock wave.



Shockwave – Bottleneck

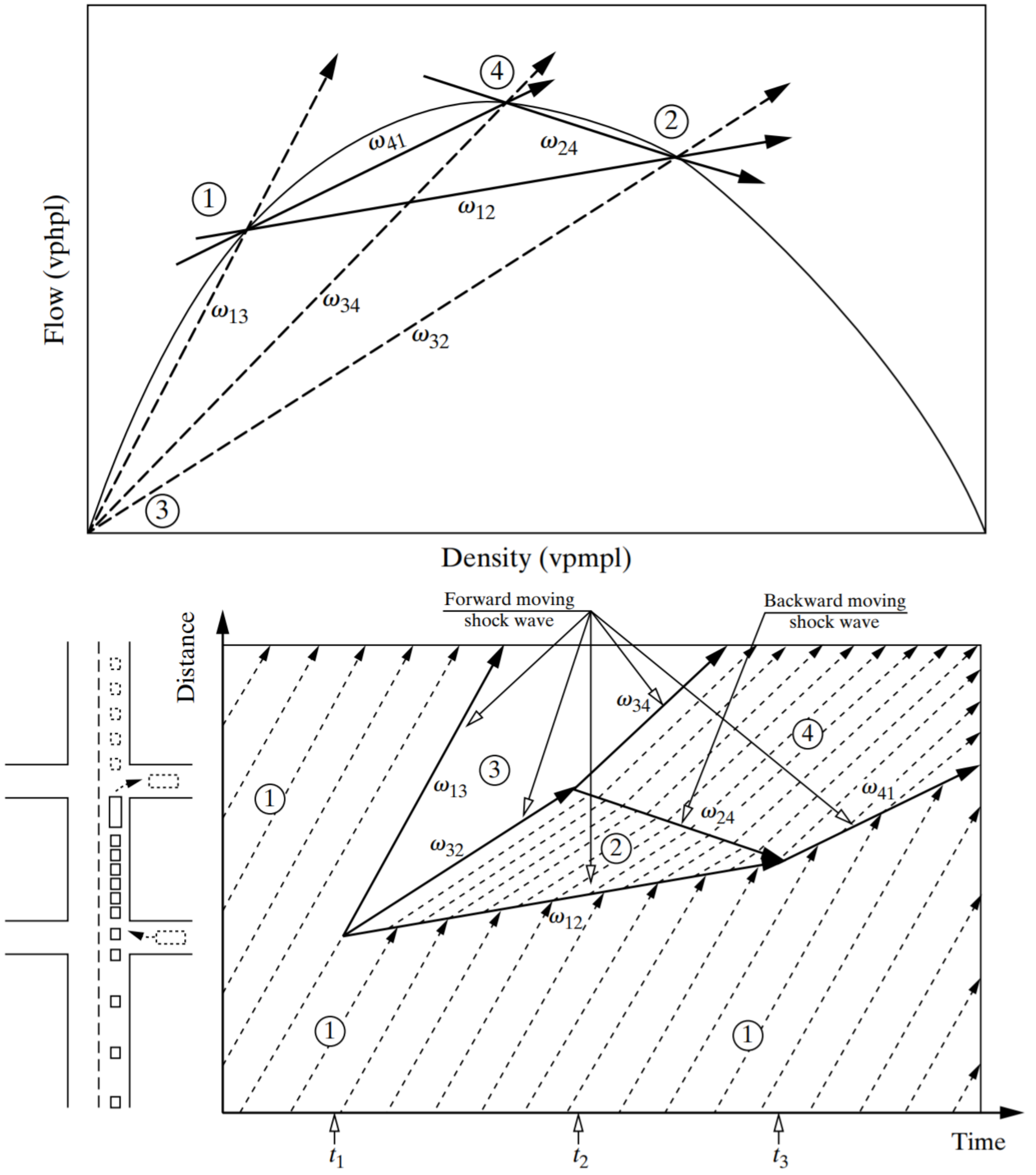
- Let us now consider the situation where the normal speed on a highway is temporarily reduced at a section of a highway where the flow is relatively high but lower than its capacity.
- For example, consider a truck that enters a two-lane highway at time t_1 and traveling at a much lower speed than the speed of the vehicles driving behind it.
- The truck travels for some time on the highway and eventually leaves the highway at time t_2.
- If the traffic condition is such that the vehicles cannot pass the truck, the shock waves that will be formed are shown in Figure.
- The traffic conditions prior to the truck entering the highway at time t_1 is depicted as section 1.



Shock Wave Created By Slow Traffic

Shockwave – Bottleneck

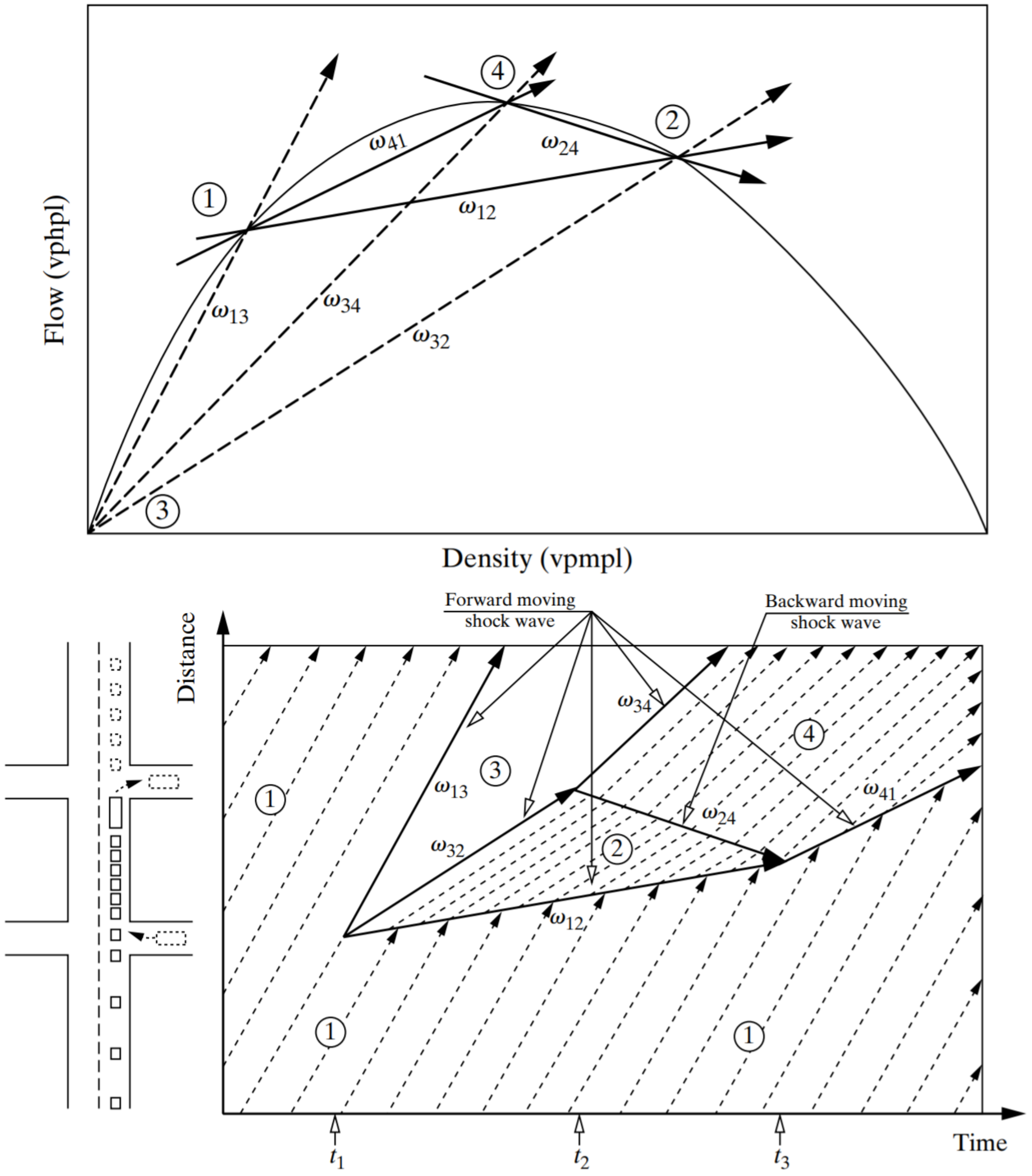
- At time t_1, vehicles immediately behind the truck will reduce their speed to that of the truck. This results in an increased density immediately behind the truck resulting in traffic condition 2 ➔ The moving shock wave with a velocity of ω_{12} is formed.
- Also, because vehicles ahead of the truck will continue to travel at their original speed, a section on the highway just downstream of the truck will have no vehicles thereby creating traffic condition 3 ➔ This also results in the formation of the forward moving shock waves with velocities of ω_{13}, and ω_{32}.



Shock Wave Created By Slow Traffic

Shockwave – Bottleneck

- At time t_2 when the truck leaves the highway, the flow will be increased to the capacity of the highway with traffic condition 4. This results in the formation of a backward moving shock wave velocity ω_{24} and a forward moving shock wave with velocity ω_{34}.
- At time t_3, shock waves with velocities ω_{12} and ω_{24} coincide resulting in a new forward moving shock wave with a velocity ω_{41}. It should be noted that the actual traffic conditions 2 and 4 depend on the original traffic condition 1 and the speed of the truck.



Shock Wave Created By Slow Traffic

Numerical Solutions

- The LWR model, the procedure for its solution, and a few concrete examples were provided to show how to apply the procedure.
- These problems were solved graphically by manually working on a time-space diagram using the method of characteristics.
- Though illustrative, the graphical approach has limitations since it can deal only with simple problems, which involve only one homogeneous highway section and simple initial conditions.
- In the real world, a traffic system may consist of a network where multiple segments or highways are considered with traffic flowing in and out via ramps.
- In addition, the initial and boundary conditions may be more complicated.
- In these cases, the graphical approach is insufficient and sometimes infeasible.

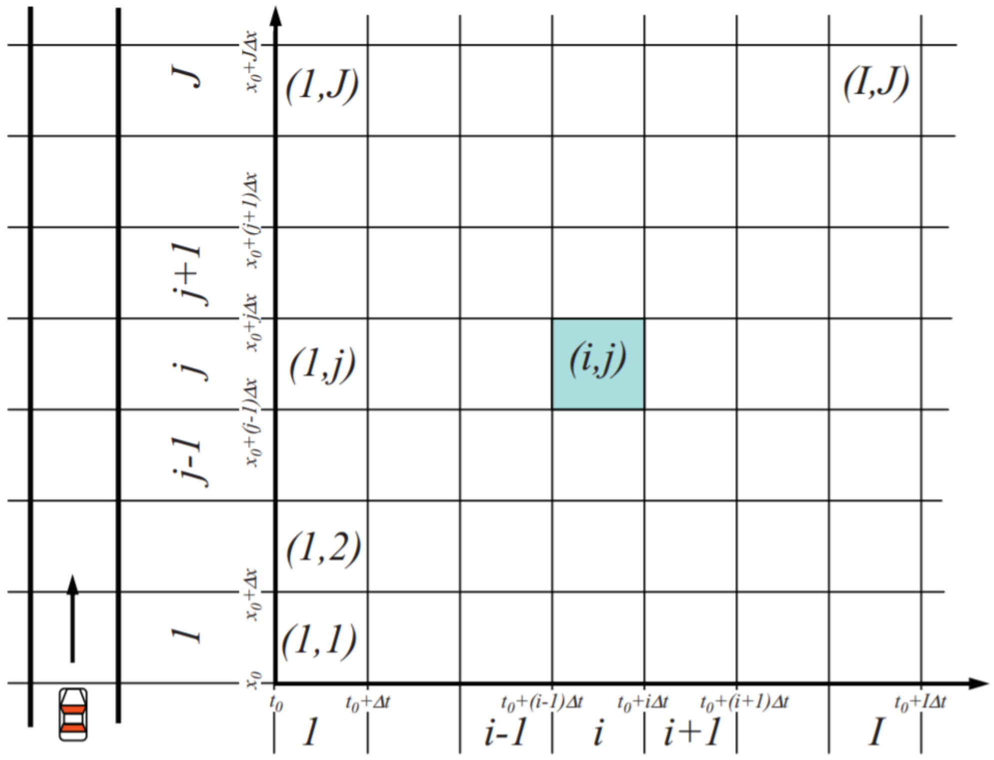
Numerical Solutions

- Moreover, the purpose of solving LWR problems is to predict traffic dynamics so that traffic engineers can anticipate congestion and to develop strategies to alleviate congestion.
- In such applications, timing is a critical issue, and solving these problems in real time is desirable.
- Moreover, the wide deployment of intelligent transportation systems makes it possible to provide real-time traffic conditions and allow online prediction.
- Therefore, a computerized solution to the LWR model is essential to cope with more complicated real-world problems, to enable real-time prediction, and to automate such predictions by the development of online applications.

Discretization Scheme

- The first step to develop a computerized solution is to discretize time and space.
- Computers are digital machines which can work only in a discrete fashion, so computerized solutions to the LWR model must be numerical and discrete.
- Figure illustrates a time-space diagram where time t is the horizontal axis and space x is the vertical axis with a roadway drawn at the side.
- The roadway is partitioned into a series of segments labeled as $j \in (0, 1, \ldots, J)$. If x_0 is chosen as the reference point and segment length x is uniform, the location of the end of segment j is:

$$x_j = x_0 + j\Delta x$$



Discretization scheme

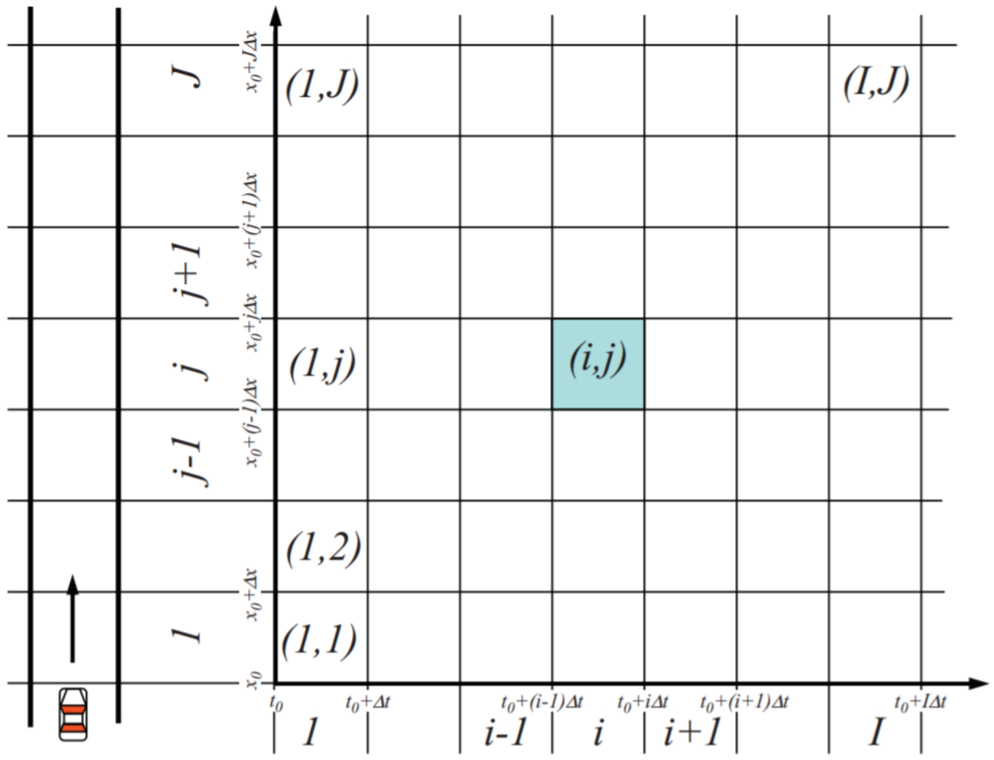
Discretization Scheme

- Similarly, the time is divided into a series of durations $i \in (0, 1, \ldots, I)$ with step size t. If the reference point of time is t_0, the end of duration i is at time:

$$t_i = t_0 + i\Delta t$$

- In general, the following relationship is required in a discretization scheme, where v_f is the free-flow speed, to ensure that a vehicle should not traverse more than one segment x within a time step t :

$$\frac{\Delta x}{\Delta t} > v_{\mathrm{f}}$$



Discretization scheme

Discretization Scheme

1. A typical numerical solution to the LWR problem starts with initial conditions by determining the number of vehicles contained in each roadway segment one by one from the upstream end to the downstream end.
2. For easy reference, the time-space region bounded within duration i and segment j is referred to as a cell and is denoted as (i, j) and the number of vehicles contained in segment j at the end of duration i is denoted as $n(t_i, x_j)$.
3. After this, time advances one step, and the above process starts over again.
4. Hence, the numerical solution consists of two loops: time t_i as the outer loop and space x_j as the inner loop.
5. The process finishes when all cells have been traversed, and the solution is given as cell storage

$$[n(t_i, x_j) | i \in (1, 2, \ldots, I), j \in (1, 2, \ldots, J)]$$

or, alternatively, traffic condition

$$k(t_i, x_j),\ q(t_i, x_j), \text{ and } v(t_i, x_j)$$

1

```
when i = 1
  determine storage in j = 1
  determine storage in j = 2
  ...
  determine storage in j = J
end
```

2

```
when i = 1
   determine n(t_1,x_1)
   determine n(t_1,x_2)
   ...
   determine n(t_1,x_J)
end
```

3

```
when i = 2
   determine n(t_2,x_1)
   determine n(t_2,x_2)
   ...
   determine n(t_2,x_J)
end
```

4

```
Numerical solution procedure:

for i = 1 to I
  for j = 1 to J
    determine n(t_i,x_j)
  end
end
```

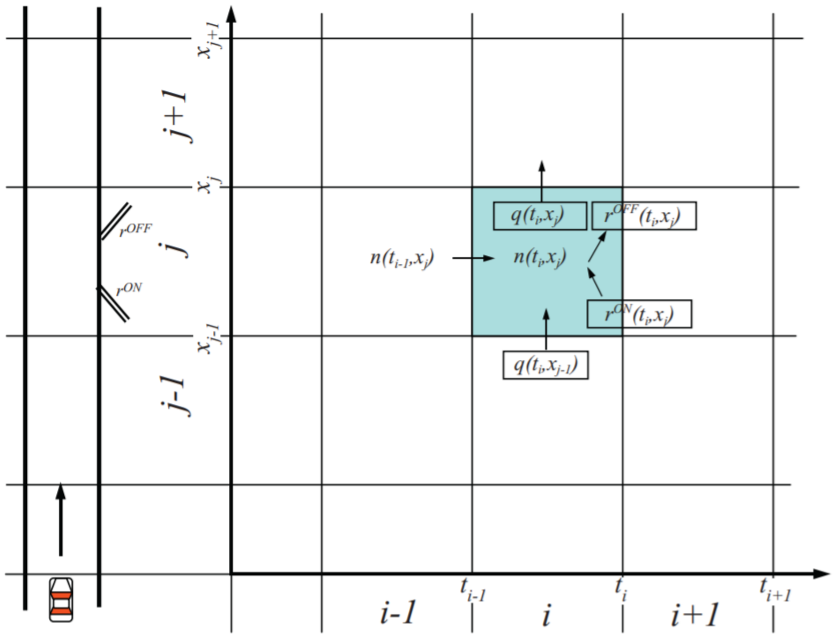
FREFLO

- FREFLO is an early (if not the earliest) computerized macroscopic traffic simulation model, developed by Payne in the late 1970s.
- Like the LWR model, FREFLO consists of three equations with a discretization scheme, shown in Figure.
- The first equation is the conservation law:
  - Storage in the current cell = Storage at previous step + Vehicles arrived from upstream - Vehicles departed to downstream + Vehicles entered via on-ramp - Vehicles exited via off-ramp
- Mathematically, this can be expressed as:

$$n(t_i, x_j) = n(t_{i-1}, x_j) + \Delta t q(t_i, x_{j-1}) - \Delta t q(t_i, x_j) + \Delta t g(t_i, x_j)$$

- where $g(t_i, x_j)$ is the net inflow via ramps—that is, $g(t_i i, x_j) = r^{on}(t_i, x_j) - r^{off}(t_i, x_j)$.
- Note that $n = k\Delta x$, and the above equation becomes

$$k(t_i, x_j) = k(t_{i-1}, x_j) + \frac{\Delta t}{\Delta x}[q(t_i, x_{j-1}) - q(t_i, x_j) + g(t_i, x_j)]$$



Discretization in FREFLO

FREFLO

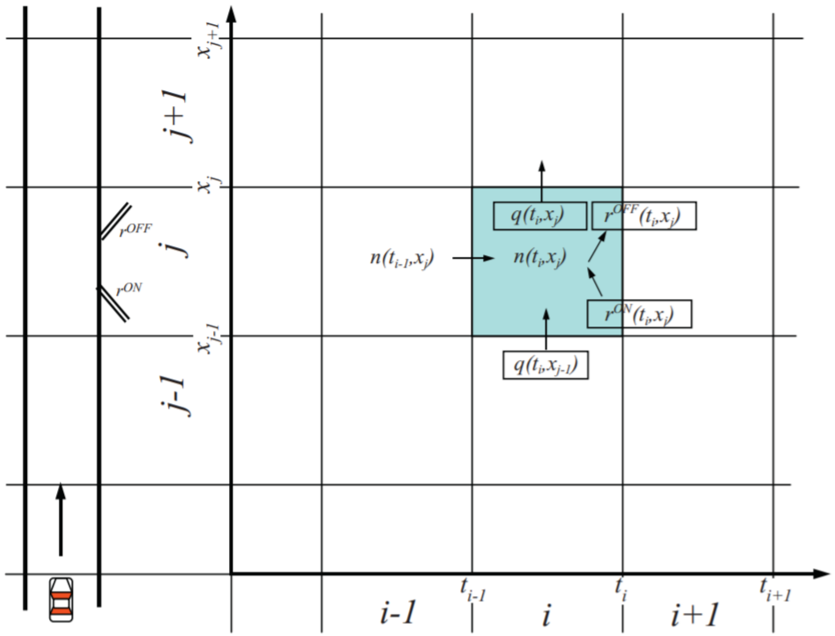
- The second equation of FREFLO is the identity in discrete form:

$$q(t_i, x_j) = k(t_i, x_j)v(t_i, x_j)$$

- For the third equation, FREFLO uses a dynamic speed-density relationship:

  speed in current cell = speed in previous step - convection + relaxation + anticipation

  - Convection - vehicles tend to continue their speeds when they travel in the upstream section,
  - Relaxation - vehicles tend to adopt the equilibrium velocity-density relationship,
  - Anticipation - vehicles tend to adjust to downstream condition, i.e. slow down if congested.



Discretization in FREFLO

FREFLO

- Mathematically, this can be expressed as:

$$v(t_i, x_j) = v(t_{i-1}, x_j) - \Delta t\{v(t_{i-1}, x_j)\frac{v(t_{i-1}, x_j) - v(t_{i-1}, x_{j-1})}{\Delta x_i} + \frac{1}{T_j}[v(t_{i-1}, x_j) - V(k(t_{i-1}, x_j)) + \frac{b_j}{k(t_{i-1}, x_j)}\frac{k(t_{i-1}, x_{j+1}) - k(t_{i-1}, x_j)}{\Delta x_j}]\}$$

$$T_j = c_T \Delta x_j$$

$$b_j = c_b \Delta x_j$$

c_T and c_b are relaxation time and anticipation coefficients, respectively

- Using observed data, the equilibrium speed-density relationship $V(k)$ takes the following form:

$$v = V(k) = \min\{88.5, (172 - 3.72k + 0.0346k^2 - 0.00119k^3)\}$$

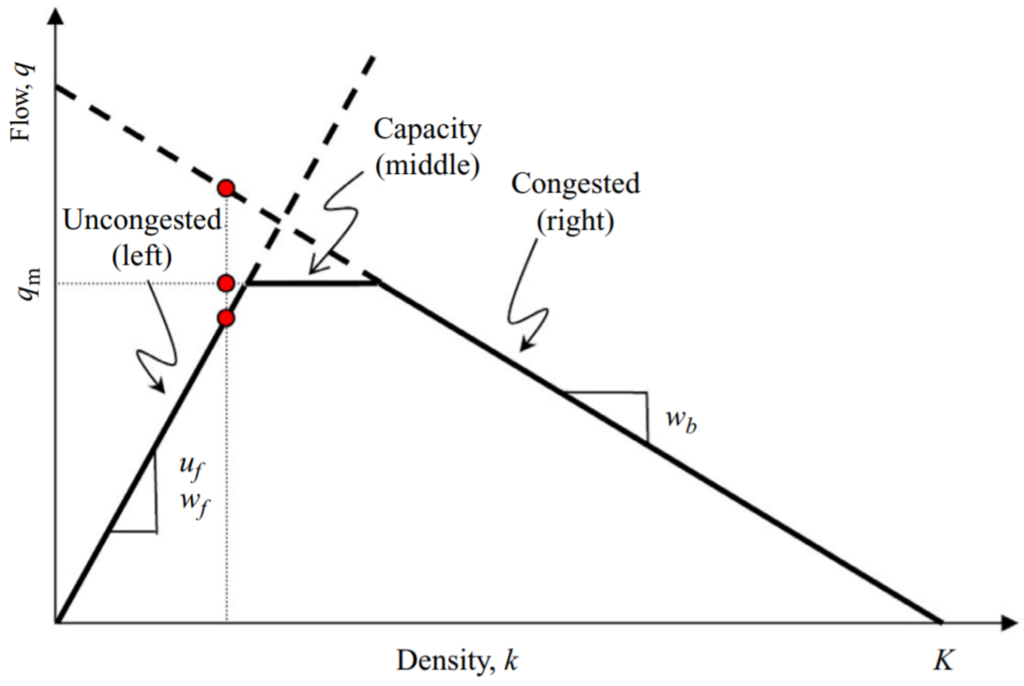
- With the above equations, one can determine the state (q, k, v) of each cell by starting from initial conditions and following the numerical solution procedure.

Cell Transmission Model

- The cell transmission model (CTM) was proposed by Daganzo in the mid-1990s.

Minimum Principle

- Figure shows a triangular flow-density relationship.
- The relationship consists of three sections:
  - Uncongested (left), with free-flow speed v_f equal to forward wave kinematic speed w_f ,
  - Capacity (middle) q_m,
  - And, congested (right), with backward wave speed w_b and jam density K.



Triangular flow-density relationship

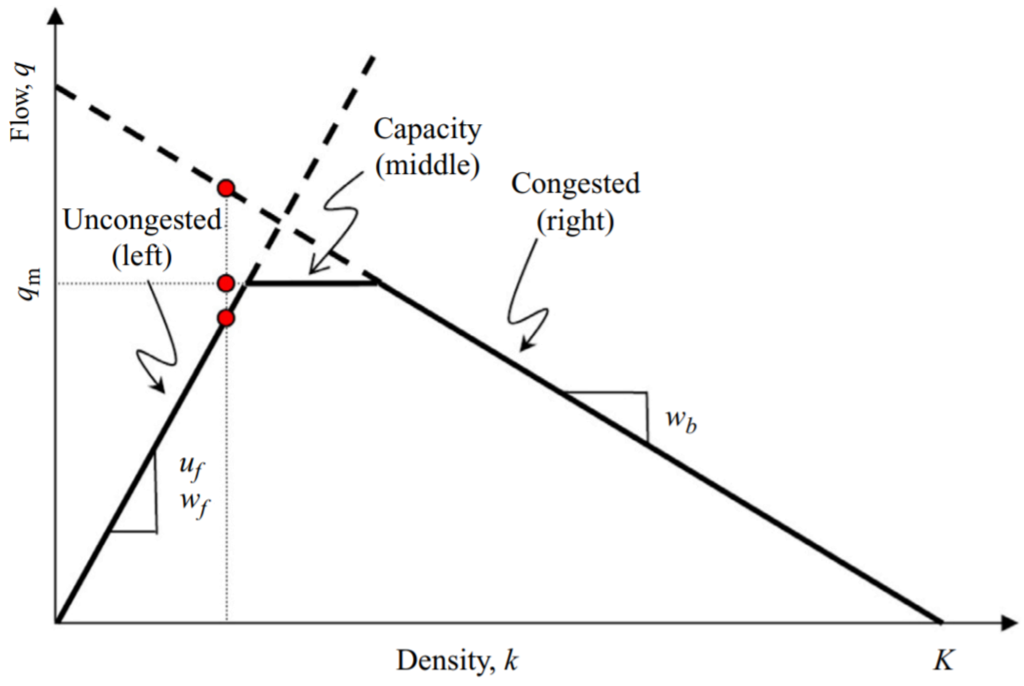
Cell Transmission Model

Minimum Principle

- A vertical line at any density k will intersect the three sections at height kw_f , q_m, and $(K - k)w_b$. Hence, flow corresponding to this density is found as the minimum of the three intersections:

$$q = \min\{kw_{\mathrm{f}}, q_{\mathrm{m}}, (K - k)w_{\mathrm{b}}\}$$

- Physically, if one considers the left section as conditions dictated by arrival traffic, the middle section as local capacity, and the right section as conditions dictated by downstream traffic, the above equation basically says that traffic flowing through a point of highway should not exceed the upstream arrival rate, local capacity, and the rate allowed by downstream conditions.



Triangular flow-density relationship

Cell Transmission Model

Mainline Scenario

- The CTM uses the same discretization scheme presented before.
- Everything else remains the same except for one thing: the cell now has a uniform length as the distance traveled by a vehicle at free-flow speed during one-time step:

$$\Delta x = v_{\mathrm{f}} \; \Delta t$$

- According to the minimum principle, traffic that can flow into segment j, $q_j \; (t_i)$, is constrained by the following:

$$q_j(t_i) = \min\{k_{j-1}(t_{i-1})w_{\mathrm{f}}, q_{\mathrm{m}}, (K - k_j(t_{i-1}))w_{\mathrm{b}}\}$$

- Hence, the number of vehicles that can move into segment j, $y_j(t_i)$, is found by multiplying both sides by Δt:

$$y_j(t_i) = q_j(t_i)\Delta t = \min\{k_{j-1}(t_{i-1})w_{\mathrm{f}}\Delta t, q_{\mathrm{m}}\Delta t, (K - k_j(t_{i-1}))w_{\mathrm{b}}\Delta t\}$$

- Note that $n \; = k\Delta x$, $x \; = \; v_f\Delta t$, and $v_f \; = w_f$ owing to the triangular flow-density relationship.

Cell Transmission Model

Mainline Scenario

- The previous equation can be transformed to the following form:

$$y_j(t_i) = \min\{k_{j-1}(t_{i-1})\Delta x, q_m \Delta t, \frac{w_b}{w_f}(K - k_j(t_{i-1}))\Delta x\}$$

$$\Downarrow$$

$$y_j(t_i) = \min\{n_{j-1}(t_{i-1}), q_m \Delta t, \frac{w_b}{w_f}(K\Delta x - n_j(t_{i-1}))\}$$

- The above equation stipulates that the number of vehicles that can move into segment j, $y_j(t_i)$, is constrained by:
  - The number of vehicles in $j - 1$ previously: $n_{j-1}\ (t_{i-1})$,
  - The capacity of segment j, $q_m \Delta t$,
  - The empty space in j: $\frac{w_b}{w_f}\ (K\Delta x\ -\ n_j\ (t_{i-1}))$

Cell Transmission Model

Mainline Scenario

- The previous equation can be further reduced to: $y_j(t_i) = \min\{S_{j-1}, R_j\}$

Flow being sent from an upstream position $S_{j-1} = \min\{n_{j-1}(t_{i-1}), q_m \Delta t\}$

Flow ready to be received downstream $R_j = \min\{q_m \Delta t, \frac{w_b}{w_f}(K\Delta x - n_j(t_{i-1}))\}$

- Therefore, the evolution of traffic on a freeway mainline can be stated as

```
Storage in current cell =
Storage in the cell previously +
Vehicles flowed in -
vehicles flowed out
```

- Mathematically, this can be expressed as:

$$n_j(t_i) = n_j(t_{i-1}) + y_j(t_i) - y_{j+1}(t_i)$$

High-Order Models

- The macroscopic traffic flow models discussed so far, including both analytical and numerical models, have been focused on the LWR model and its variants.
- At the center of these models is mass or vehicle conservation, which can be mathematically expressed as a first-order partial differential equation:

$$\frac{\partial k(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0$$

  where k and q are density and flow, which depend on time t and space x.
- Hence, these models are referred to as *first-order* models.
- Common to first-order models is their prediction of a shock wave when two kinematic waves meet.
- Consequently, a vehicle crossing the shock wave has to change its speed abruptly, which is physically impossible.
- Efforts to address these undesirable features has led many researchers to seek more realistic models to represent traffic dynamics.
- These efforts gave rise to high-order dynamic traffic flow models.

High-Order Models

- Daganzo noted that, the LWR model, which is a first-order continuum flow model, is proposed for dense traffic with an equilibrium and it is flawed for traffic at light traffic conditions.
- When passing is allowed, the LWR model produces unsatisfactory results in the following aspects:
  - The LWR model predicts an abrupt speed change when a vehicle passes through a shock wave, an action that is unrealistic in the real world.
  - The LWR model fails to predict instabilities of stop-start traffic.
  - The LWR model assumes zero reaction time, which does not happen in the real world.
- These shortcomings imply that when passing is allowed, the LWR model fails to recognize that the preferred speed for each vehicle varies over time and the desired speeds among a group of vehicles vary as well.
- These variations can cause a platoon to disperse in a way that is not predicted by the LWR model.

High-Order Models

- Given these deficiencies, the continuum flow models developed so far have been trying to fix the deficiencies, and almost all of these models follow the direction of incorporating a momentum conservation equation.
- Payne and Whitham proposed a dynamic model, the so-called PW model (1971), trying to smooth out the discontinuity in speed change across shock waves.
- A momentum equation was introduced in this model to describe the structure of a shock wave.
- This seminal work has inspired many future works.

PW Model (1971)

- Proposed by Payne and independently by Whitham, the PW model consists of a system of two equations:
  - The first is the conservation of mass given in the LWR model,
  - The second equation is derived from the Navier-Stokes equation of motion for a one-dimensional compressible flow with a pressure and a relaxation term.

$$\begin{cases} \frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0, \\ \frac{\partial v}{\partial t} = -v\frac{\partial v}{\partial x} - \lambda(v - V_\mathrm{d}(k)) - \frac{1}{k}\frac{\mathrm{d}P}{\mathrm{d}k}\frac{\partial k}{\partial x} \end{cases}$$

  - where v is traffic speed, $V_d(k)$ is the equilibrium speed-density relationship, $P(k)$ is traffic pressure, and λ is a coefficient.
- The momentum equation in this model describes the structure of a shock wave.
- This equation tries to smooth out the discontinuity in speed change across shock waves.
- Note that FREFLO (presented previously) is a numerical solution to the PW model.

PW Model (1971)

- Several deficiencies are found in the PW model:
    1. It does not remove all the shock waves.
    2. Vehicles in the PW model can adjust their speeds in response to disturbance from behind, while in reality, vehicles typically respond to their leaders.
    3. The PW model incorporates a momentum equation, which is derived from a car-following model. This momentum equation does not consider second-order and higher-order terms of spacings and speeds, which may not be negligible when spacings and speeds are not slowly varying.
    4. The PM model as well as other high-order models always produces wave speeds that are greater than traffic speeds. This is an unattractive property of macroscopic models because it implies that future conditions of a vehicle are partially decided by what happens behind it.

PW Model (1971)

- Several deficiencies are found in the PW model:
  5. The strength that high-order models smooth out shocks turns out to be these models' weakness. This is because any model that attempts to smooth all the discontinuities must sometimes predict negative speeds and such negative speeds cannot be removed by convergent numerical approximation methods.
  6. Sixth, but probably not the last, high-order models involve more complex partial differential equations and more variables, which increases computational complexity, and are more difficult to calibrate and implement.
- Given these limitations, many researchers tend to believe that high-order models, despite their added complexity and additional parameters, might not be superior to the LWR model.

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Thank
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