Frequency Domain Filtering of Colour Images using Quaternion Fourier Transforms

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Abstract
The 2D Fourier Transform is an important image processing tool to decompose a grayscale image into its sine and cosine components. The output of the transformation represents the image in the frequency domain, given the input image in the spatial domain. Filtering in the frequency domain is a common image and signal processing technique. In this paper, we propose a new approach for filtering of colour images in the frequency domain using quaternion Fourier transform(also known as hypercomplex Fourier transform). This transform allows colour images to be transformed as a whole, rather than as colour separated components. Specifically, this paper deals with colour image smoothing and sharpening in the frequency domain by using quaternion Fourier transform. The experimental results indicate the effectiveness of this method in colour image filtering.

Keywords
Colour image smoothing, colour image sharpening, quaternions, quaternion Fourier transforms.

I. Introduction

Fourier transforms have been widely used in image processing applications, such as image analysis, image filtering, image reconstruction and image compression. Until recently, there was no definition of a Fourier transform applicable to colour images in a holistic manner. The idea of computing the Fourier transform of a colour image has only recently been realized. It is possible to separate a colour image into three scalar images and compute the Fourier transforms of these images separately. In this paper we are concerned with the computation of a single, holistic, Fourier transform which treats a colour image as a vector field and application of this transform to colour image filtering such as lowpass filtering for smoothing and highpass filtering for sharpening.

The first definition of a quaternion Fourier transform was that of Ell [1] and the first application of a quaternion Fourier transform of colour images was reported in 1996 [2] using a discrete version of Ell' transform. In this work, we use the more recent quaternion Fourier transform definition [3]. The application of a quaternion Fourier transform to colour images is based on representing colour image pixels using quaternions discovered by Hamilton in 1843 [4].

II. Quaternion numbers

The concept of the quaternion was introduced by Hamilton in 1843 [4]. It is the generalization of a complex number. A complex number has two components: the real and the imaginary part. The quaternion, however, has four components, i.e., one real part and three imaginary parts and can be represented in Cartesian form as:

$$q = w + xi + yj + zk \qquad (1)$$

where w, x, y and z are real numbers and i, j and k are complex operators which obey the following rules.

$$ij = k,\ \ jk = i,\ \ ki = j,$$

$$ji = -k, kj = -i, ik = -j$$

and also satisfies $i^2 = j^2 = k^2 = ijk = -1$. From these rules, it is clear that multiplication is not commutative. The quaternion conjugate is $\bar{q} = w - xi - yj - zk$ (2)

and the modulus of a quaternion is given by

$$|q| = \sqrt{w^2 + x^2 + y^2 + z^2} \qquad (3)$$

A quaternion with zero real part is called a pure quaternion and a quaternion with unit modulus is called a unit quaternion. The imaginary part of a quaternion has three components and may be associated with a 3-space vector. For this reason, it is sometimes useful to consider the quaternion as composed of a vector part and a scalar part .Thus q can be expressed as

$$q = S(q) + V(q), \qquad (4)$$

where the scalar part ,$S(q)$is the real part i.e. S(q)=w and the vector part is a composite of three imaginary components,

$$V(q) = xi + yj + zk.$$

III. Quaternion Representation of Color Image Pixels

Color image pixels have three components, and they can be represented in quaternion form using pure quaternions. For images in RGB colour space, the three imaginary parts of a pure quaternion can be used to represent the red, green and blue colour components. For example, a pixel at image coordinates (x, y) in an RGB image can be represented as

$$f(x,y) = r(x,y)i + g(x,y) + b(x,y) \qquad (5)$$

where $r(x,y)$, $g(x,y)$and $b(x,y)$ are the red, green and blue components of the pixel, respectively.
Using quaternions to represent the RGB color space, the three color channels are processed equally in operations such as multiplication. The advantage of using quaternion based operations to manipulate color information in an image is that we do not have to process each colour channel independently, but rather, treat each color triple as a whole unit. We believe, by using quaternion operations, higher color information accuracy can be achieved because a color is treated as an entity.

IV. Quaternion Fourier Transform

Based on the concept of quaternion multiplication and exponential, the Quaternion Fourier Transform (QFT) has been introduced. Due to the non-commutative property of the quaternion, there are three different types of QFT defined: the left side QFT, the right side QFT and the two sides QFT. The earliest definition of QFT is the two-side form as following [1].

$$F_q(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iux} f(x,y) e^{-jvy}\,\mathrm{dxdy} \quad (6)$$

In fact, the QFT defined above can be generalized as [1]

$$F_{L-R}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mu_1 ux} f(x,y) e^{-\mu_2 vy}\,\mathrm{dxdy} \quad (7)$$

where μ_1 and μ_2 are two unit pure quaternions (i.e., the quaternions with unit magnitude and zero real part) that are orthogonal to each other.

More recently, the left-side the right-side form of QFT were defined in [2] as

Left-Side QFT:

$$F_L(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mu_1(ux+vy)} f(x,y)\,\mathrm{dxdy} \quad (8)$$

Right-Side QFT:

$$F_R(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\; e^{-\mu_1(ux+vy)}\,\mathrm{dxdy} \quad (9)$$

Similarly, the Inverse Quaternion Fourier Transforms (IQFT) can be defined for the three types of QFT respectively as [2]:

Two-Sides IQFT:

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mu_1 ux} F_{L-R}(u,v) e^{-\mu_2 ux}\,dudv \quad (10)$$

Left-Side IQFT:

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\mu_1(ux+vy)}\; F_L(u,v)\,dudv \quad (11)$$

Right-Side IQFT:

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_R(u,v)\, e^{-\mu_1(ux+vy)}\,dudv \quad (12)$$

In the discrete case, the discrete quaternion Fourier transforms (DQFT) and discrete Inverse Quaternion Fourier transforms (IDQFT) are also defined in these three types [2]. Discrete version of the Left-Side, Right-side, and Two-side quaternion Fourier transforms can be represented as

Left-side DQFT (Type 2):

$$F_L(u,v) = \frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} e^{-\mu 2\pi\left(\frac{xu}{M}+\frac{yv}{N}\right)} f(x,y) \quad (13)$$

Right-side DQFT (Type 3):

$$F_R(u,v) = \frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\, e^{-\mu 2\pi\left(\frac{xu}{M}+\frac{yv}{N}\right)} \quad (14)$$

Two-side DQFT (Type1):

$$F_{L-R}(u,v) = \frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} e^{-\mu 2\pi\frac{xu}{M}} f(x,y)\, e^{-\mu 2\pi\frac{vy}{N}} \quad (15)$$

Similarly, the Inverse Discrete Quaternion Fourier Transforms (IDQFT) can be defined for the three types of QFT respectively as

Left-side IDQFT:

$$f(x,y) = \frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} e^{\mu 2\pi\left(\frac{xu}{M}+\frac{yv}{N}\right)} F_L(u,,v) \quad (16)$$

Right-side IDQFT:

$$f(x,y) = \frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} F_R(u,v)\, e^{\mu 2\pi\left(\frac{xu}{M}+\frac{yv}{N}\right)} \quad (17)$$

Two-side IDQFT:

$$f(x,y) = \frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} e^{\mu 2\pi\frac{xu}{M}} F_{L-R}(u,v)\, e^{\mu 2\pi\frac{vy}{N}} \quad (18)$$

In this transform, the hypercomplex operator was generalized: μ is any unit pure quaternion. μ determines a direction in color space and an obvious choice for color images is the direction corresponding to the luminance axis which connects all the points r=g=b. In RGB color space this is the gray line.

V. Filtering in the Frequency Domain

The frequency domain is nothing more than the space defined by values of the Fourier Transform and its frequency variables (u, v). In this section, we define different types of quaternion smoothing and sharpening filters. In the case where $h(x,y)$ (impulse response of quaternion filter) has the even symmetry relation

$$h(x,y) = h(-x,-y) \quad (19)$$

then we can prove that the type 2 QFT of $h(x,y)$ denoted by $H_q(u,v)$ also has the same symmetry relation and the relation between the quaternion convolution and the type 2 QFT can be simplified to $G(u,v) = H_q(u,v)F(u,v)$.

So, when $h(x,y) = h(-x,-y)$, the quaternion convolution operation in the spatial domain corresponds to the product operation in the frequency domain [5]. This is the same as the case of the conventional convolution. Our basic model of filtering in the frequency domain is given by

$$G(u,v) = H_q(u,v)\, F(u,v) \quad (20)$$

where F(u,v) is the type 2 quaternion Fourier transform of the color image to be filtered and G(u,v) is the type 2 quaternion Fourier transform of the filtered output image. The objective is to select a quaternion filter transfer function H_q H_q (u,v) that yields G(u,v).The filtered image is obtained simply by taking the inverse quaternion Fourier transform of G (u,v).

1. Smoothing frequency domain quaternion filters.

The edges and other sharp transitions in the pixels of an image contribute significantly to the high frequency content of its Fourier Transform. Hence, smoothing of a colour image is achieved in the frequency domain by attenuating the specified range of high frequency components in the quaternion Fourier transform of the image. Now we propose three types of quaternion low pass filters whose impulse responses satisfy Eq. (19): ideal, Butterworth, and Gaussian filters [7].

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2. Ideal quaternion low pass filters

The simplest lowpass quaternion filter is a filter that cuts off all high frequency components of the quaternion Fourier Transform that are at a distance greater that a specified distance Do from the origin of the transform. Such a filter has the transfer function

$$H_q(u,v) = \begin{cases} 1, & D(u,v) \le D_0 \\ 0, & D(u,v) > D_0 \end{cases} \quad (21)$$

where D0 is a non-negative quantity, and D(u,v) is the distance from point (u,v) to the origin of the frequency rectangle (M/2, N/2), given by

$$D(u,v) = \left[\left(u - \frac{M}{2}\right)^2 + \left(v - \frac{N}{2}\right)^2 \right]^{\frac{1}{2}} \quad (22)$$

Fig.1 shows the results of ideal lowpass filtering in the hypercomplex spectral domain. In the first column of Fig.1, are the original image and its spectral modulus. The second, third, and fourth columns, in the same Fig. , show the results of ideal lowpass filtering the image by masks with radii 10, 20 and 30. As expected the resulting images are blurred consistent with this operation. As the filter radius increases, less and less power is removed, resulting in less blurring.

2. Butterworth quaternion lowpass filters

The transfer function of a Butterworth quaternion lowpass filter of order n, and with cutoff frequency at a distance D0 from the origin, is defined as

$$H_q(u,v) = \frac{1}{1+\left[\frac{D(u,v)}{D_0}\right]^{2n}} \quad (23)$$

where D(u,v) is given by Eq.(22). Fig.2 shows the results of Butterworth lowpass filtering the Lena image in the hypercomplex spectral domain. Unlike the results shown in Fig.1 for the ideal lowpass filter, we note here a smooth transition in blurring as a function of increasing cutoff frequency.

3. Gaussian quaternion lowpass filters

The two dimensional Gaussian lowpass filter transfer function with the cutoff frequency at a distance D0 is given by

$$H_q(u,v) = e^{-\frac{D^2(u,v)}{2\sigma^2}} \quad (24)$$

where σ is a measure of Gaussian spread and is equal to D0 and D(u,v) as in Eq.(22) is the distance from the origin of the Fourier transform. Fig.3 shows the results of Gaussian quaternion lowpass filtering the Lena image in the hypercomplex spectral domain. As in the case of the Butterworth lowpass filter, we note a smooth transition in blurring as a function of increasing cutoff frequency. The Gaussian lowpass filter did not achieve as much smoothing as the Butterworth lowpass filter of order 2 for the same value of cutoff frequency.

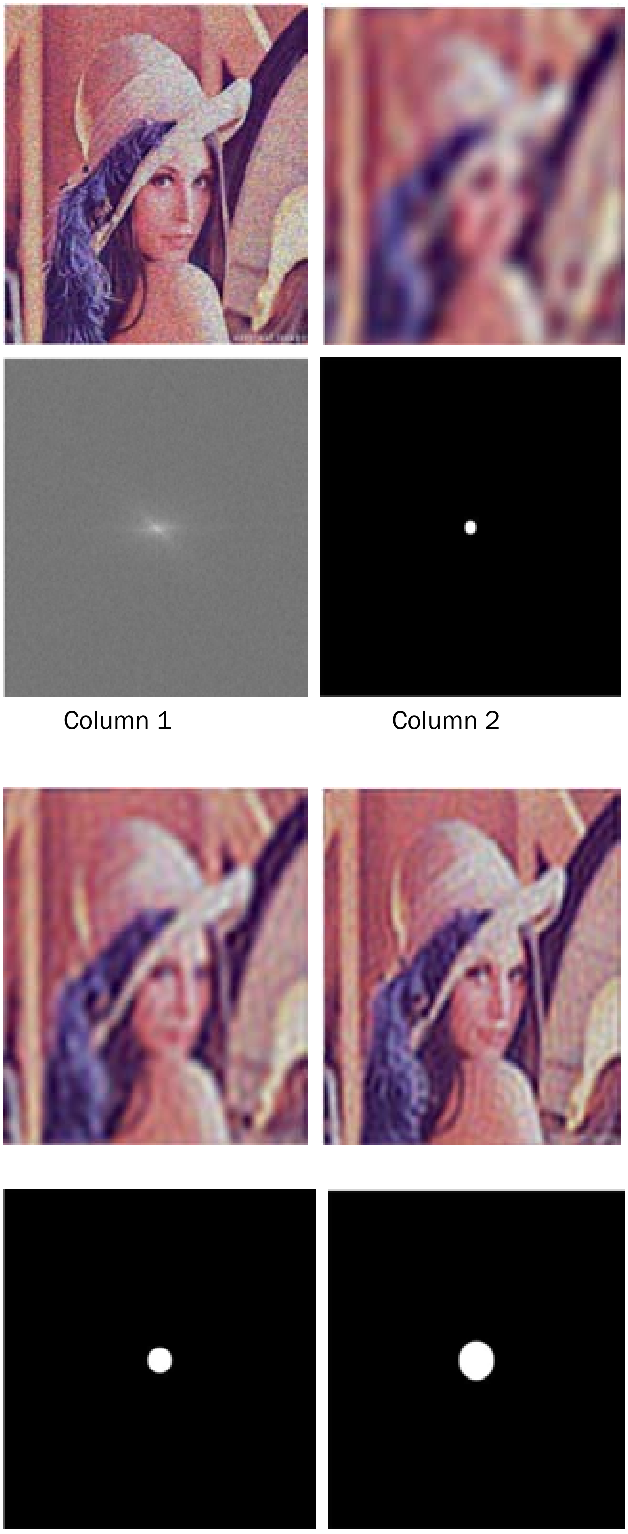

Fig. 1: The results of ideal lowpass filtering based on type 2 QFT, first column shows the original noisy colour image and its spectral modulus with the axis $\mu_1 = [1 \;\; 1 \;\; 1]/\sqrt{3}$ and $\mu_2 = [0 \;\; 1 -1]/\sqrt{2}$. The lowpass filtering of the image by the masks with radii 10,20 and 30 and the corresponding type two inverse QFT(Left-Side) images are likewise shown in the second ,third and fourth columns, respectively.

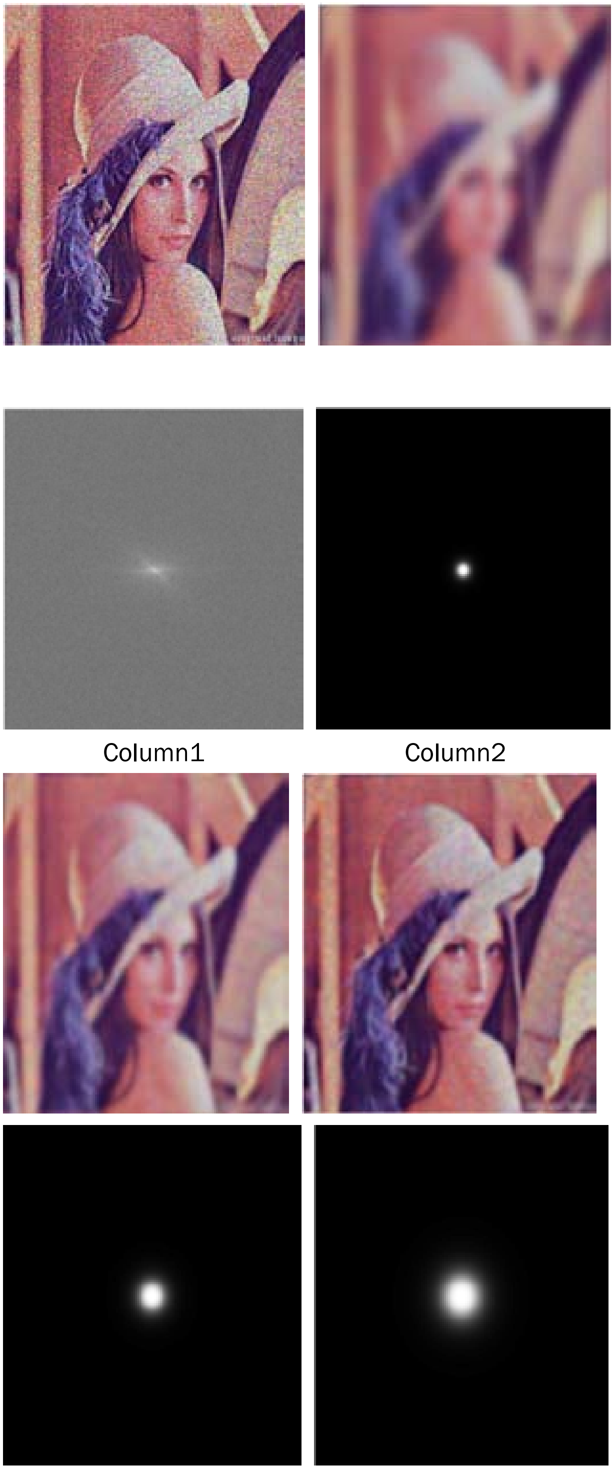

Fig. 2: The results of Butterworth lowpass filtering of order 2 based on type 2 QFT, first column shows the original noisy colour image and its spectrum modulus with the axis $\mu_1 = [1 \ \ 1 \ \ 1]/\sqrt{3}$ and $\mu_2 = [0 \ \ 1 - 1]/\sqrt{2}$. The lowpass filtering of the image by the masks with radii 10,20 and 30 and the corresponding type two inverse QFT(Left-Side) images are likewise shown in the second ,third and fourth columns respectively.

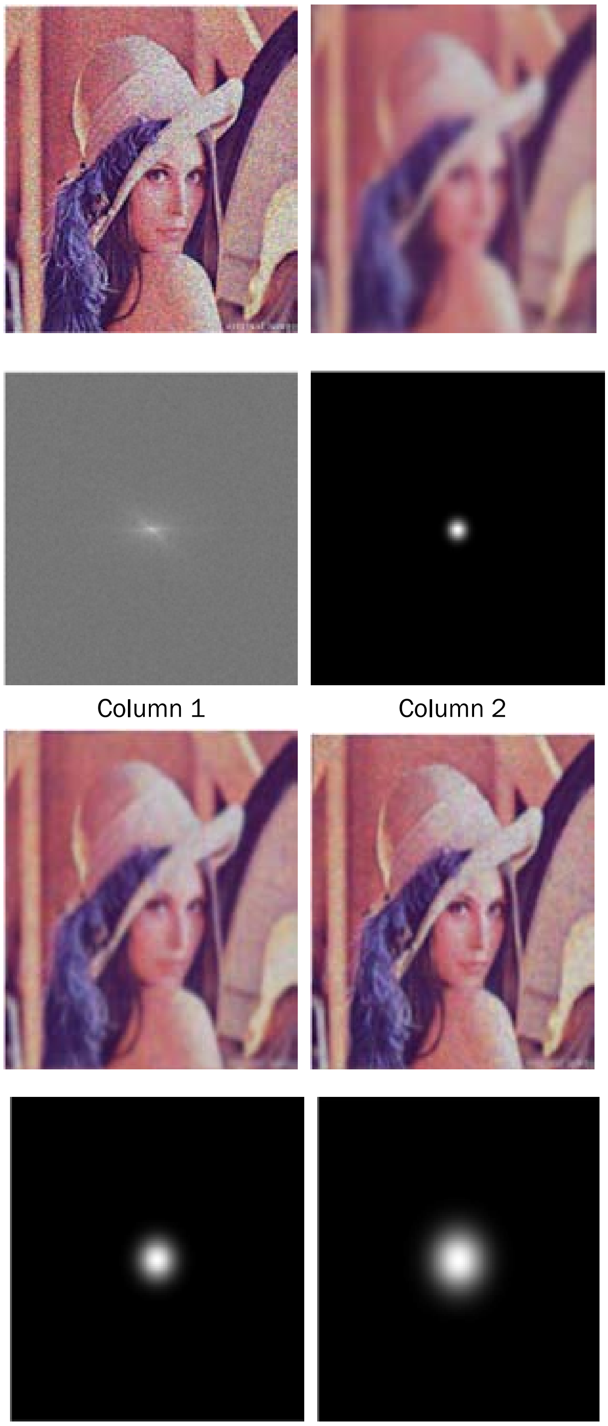

Fig. 3: The results of Gaussian lowpass filtering based on type 2 QFT, first column shows the original noisy colour image and its spectrum modulus with the axis $\mu_1 = [1 \ \ 1 \ \ 1]/\sqrt{3}$ and $\mu_2 = [0 \ \ 1 - 1]/\sqrt{2}$. The lowpass filtering of the image by the masks with radii 10,20 and 30 and the corresponding type two inverse QFT(Left-Side) images are likewise shown in the second ,third and fourth columns, respectively.

B. Sharpening frequency domain quaternion filters

The edges and other abrupt changes in pixels are associated with high frequency components, image sharpening can be achieved in the frequency domain by a high pass filtering process, which attenuates the low frequency components without disturbing high frequency information in the quaternion Fourier transform.

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Because the intended function of the quaternion highpass filter is to perform the reverse operation of the quaternion lowpass filter, the transfer function of the highpass filters can be obtained using the relation:

$$H_{hpq}(u,v) = 1 - H_{lpq}(u,v) \qquad (25)$$

where $H_{lpq}(u,v)H_{lpq}(u,v)$ is the transfer function of the corresponding lowpass quaternion filter. In this section also, we propose three types of quaternion highpass filters that satisfy Eq. (19): ideal, Butterworth, and Gaussian [7].

1. Ideal highpass quaternion filters

A two dimensional quaternion highpass filter is defined as

$$H_q(u,v) = \begin{cases} 0, & D(u,v) \le D_0 \\ 1, & D(u,v) > D_0 \end{cases} \qquad (26)$$

where D0 is a non-negative quantity, and D(u, v) is the distance from point (u, v) to the origin of the frequency rectangle, as given by Eq.(22).

Fig.4 shows the results of ideal highpass filtering in the hypercomplex spectral domain. In the first column, are the original image and its spectral modulus. The second, third, and fourth columns, in the same Fig. , show the results of highpass filtering the image by masks with radii 5, 10 and 20. Again, as expected, the image contains content where there is rapid luminance and chrominance variation. From this Fig. it can be seen that the sharpening of the image increases as radius of mask increases.

2. Butterworth highpass quaternion filters

The transfer function of the Butterworth quaternion highpass filter of order 'n' and with cutoff frequency locus at a distance D0 from the origin is given by

$$H_q(u,v) = \frac{1}{1+\left[\frac{D_0}{D(u,v)}\right]^{2n}} \qquad (27)$$

where D(u,v) is the distance as given by Eq. (22).

Fig.5 shows the results of Butterworth highpass filtering the Lena image in the hypercomplex spectral domain. In the first column, are the original image and its spectral modulus. The second, third, and fourth columns, in the same Fig. , show the results of Butterworth highpass filtering the image by masks with radii 5, 10 and 20. As in the case of lowpass filters, we can expect Butterworth highpass filters to behave smoother than ideal highpass filters.

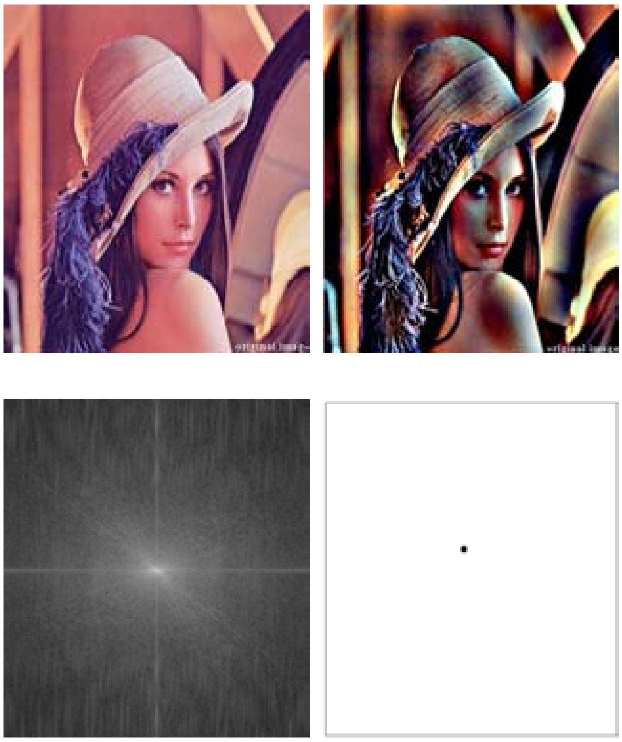
3. Gaussian highpass quaternion filters

The transfer function of the Gaussian highpass quaternion filter with cutoff frequency locus at a distance D0 from the origin is given by

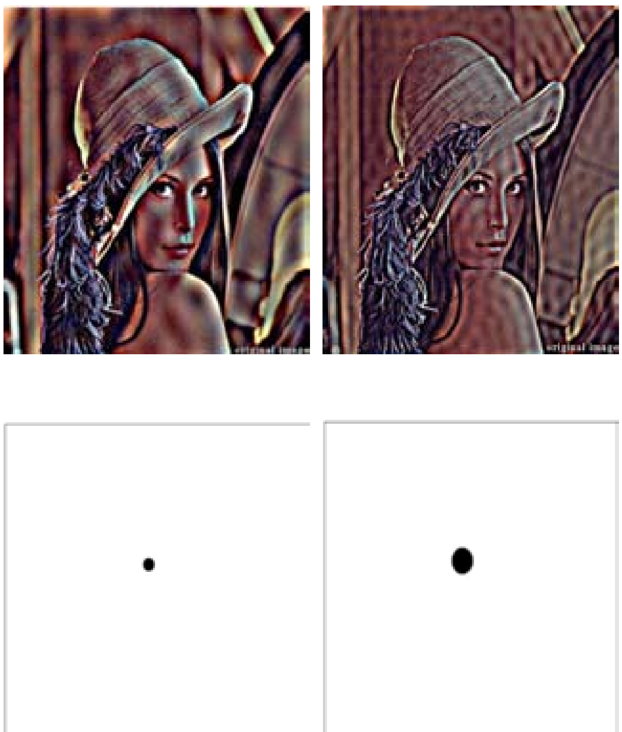
$$H_q(u,v) = 1 - e^{-\frac{D^2(u,v)}{2\sigma^2}} \qquad (28)$$

where σ is a measure of Gaussian spread and is equal to D0 and D(u,v) is the distance as given by Eq. (22).

Fig.6 shows the results of Gaussian highpass filtering the Lena image in the hypercomplex spectral domain. In the first column, are the original image and its spectral modulus. The second, third, and fourth columns, in the same Fig. , show the results of Gaussian highpass filtering the image by masks with radii 5, 10 and 20. The results obtained with Gaussian highpass filtering are smoother than with ideal highpass filters and Butterworth highpass filters.

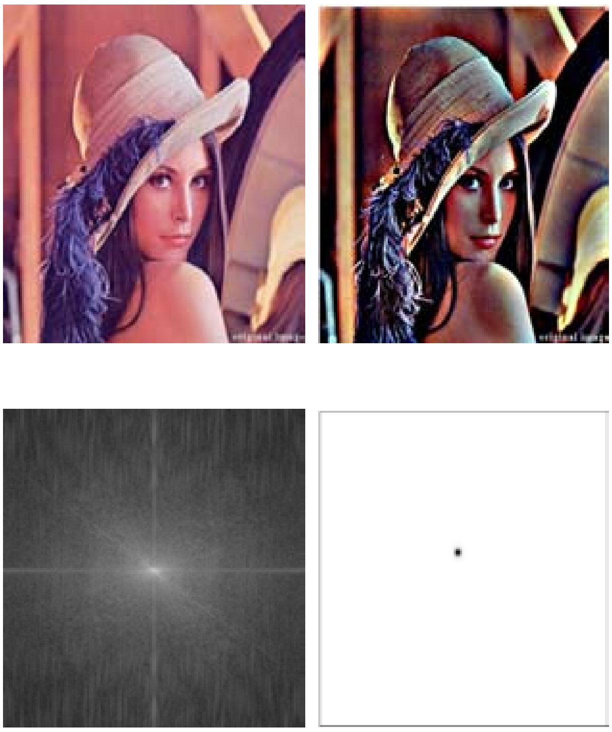


Column 1 Column 2



Column 3 Column 4

Fig. 4: The results of ideal highpass filtering based on type 2 QFT, first column shows the original colour image and its spectrum modulus with the axis $\mu_1 = [1 \;\; 1 \;\; 1]/\sqrt{3}$ and $\mu_2 = [0 \;\; 1 - 1]/\sqrt{2}$. The highpass filtering of the image by the masks with radii 5,10 and 20 and the corresponding type two inverse QFT(Left-Side) images are likewise shown in the second ,third and fourth columns, respectively.



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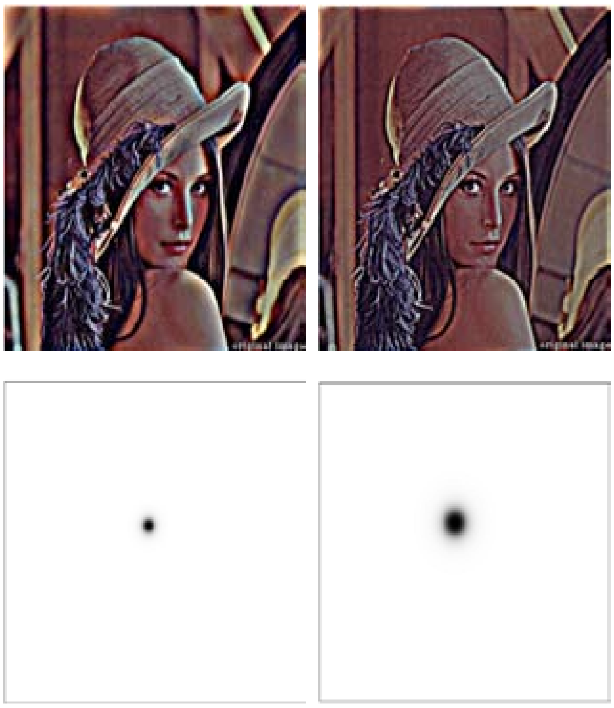
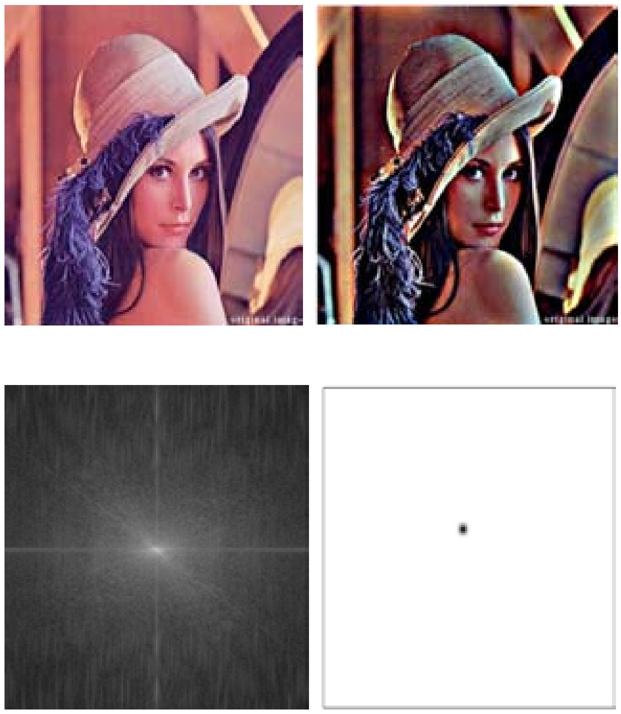
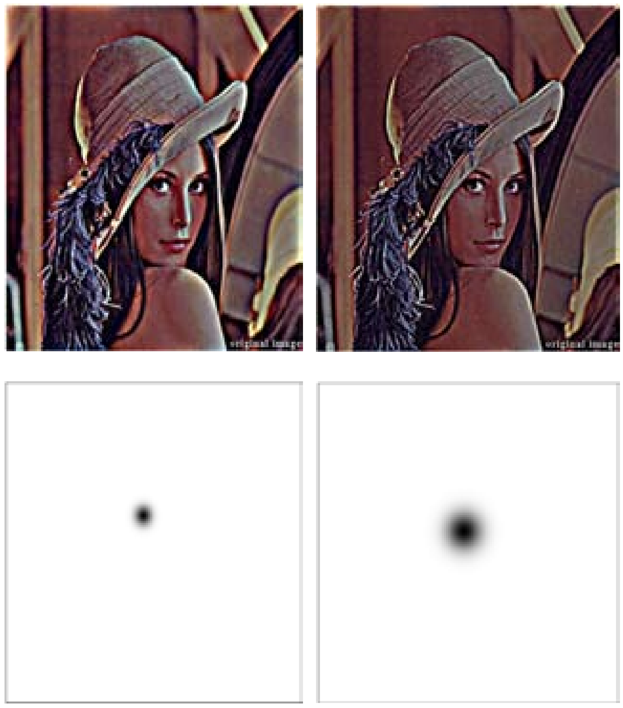


Column 3 Column 4

Fig. 5: The results of Butterworth highpass filtering of order 2 based on type 2 QFT, first column shows the original colour image and its spectrum modulus with the axis $\mu_1 = [1 \;\; 1 \;\; 1]/\sqrt{3}$ and $\mu_2 = [0 \;\; 1 - 1]/\sqrt{2}$. The highpass filtering of the image by the masks with radii 5,10 and 20 and the corresponding type two inverse QFT(Left-Side) images are likewise shown in the second ,third and fourth columns, respectively.



Column 1 Column 2



Column 3 Column 4

Fig. 6: The results of Gaussian highpass filtering based on type 2 QFT, first column shows the original colour image and its spectrum modulus with the axis $\mu_1 = [1 \;\; 1 \;\; 1]/\sqrt{3}$ and $\mu_2 = [0 \;\; 1 - 1]/\sqrt{2}$. The highpass filtering of the image by the masks with radii 5,10 and 20 and the corresponding type two inverse QFT(Left-Side) images are likewise shown in the second ,third and fourth columns, respectively.

VI. Conclusion

This work demonstrates that the Quaternion Fourier Transform is well suited to describing the spectral content of colour images. Similar to gray scale images, colour images represented as quaternion valued images can also be transformed into the Fourier domain and can be represented as quaternion frequency signals, based on which different image processing techniques such as filtering can be performed efficiently. Filtering in quaternion frequency domain has the advantage that the colour triples are processed as a whole unit rather than dealing with RGB channels separately. We believe more accurate colour information can be preserved this way, since all colour channels are processed as a single unit. And we would like to point out that, the Quaternion Fourier transform is not limited to this, but also can be applied to other colour image processing fields, such as image registration, edge detection, and data compression.

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