Chapter 3

The Construction of a Finite Element Space

To approximate the solution of the variational problem,

$$a(u,v) = F(v) \quad \forall\, v \in V,$$

developed in Chapter 0, we need to construct finite-dimensional subspaces $S \subset V$ in a systematic, practical way.

Let us examine the space S defined in Sect. 0.4. To understand fully the functions in the space S, we need to answer the following questions:

1. What does a function look like in a given subinterval?
2. How do we determine the function in a given subinterval?
3. How do the restrictions of a function on two neighboring intervals match at the common boundary?

In this chapter, we will define piecewise function spaces that are similar to S, but which are defined on more general regions. We will develop concepts that will help us answer these questions.

3.1 The Finite Element

We follow Ciarlet's definition of a finite element (Ciarlet 1978).

(3.1.1) Definition. *Let*

(*i*) $K \subseteq \mathbb{R}^n$ *be a bounded closed set with nonempty interior and piecewise smooth boundary* (*the* **element domain**),

(*ii*) $\mathcal{P}$ *be a finite-dimensional space of functions on* K (*the space of* **shape functions**) *and*

(*iii*) $\mathcal{N} = \{N_1, N_2, \ldots, N_k\}$ *be a basis for* $\mathcal{P}'$ (*the set of* **nodal variables**).

Then $(K, \mathcal{P}, \mathcal{N})$ *is called a* **finite element**.

It is implicitly assumed that the nodal variables, N_i, lie in the dual space of some larger function space, e.g., a Sobolev space.

(3.1.2) Definition. *Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element. The basis $\{\phi_1, \phi_2, \ldots, \phi_k\}$ of $\mathcal{P}$ dual to $\mathcal{N}$ (i.e., $N_i(\phi_j) = \delta_{ij}$) is called the* **nodal basis** *of $\mathcal{P}$.*

(3.1.3) Example. (the 1-dimensional Lagrange element) Let $K = [0,1]$, $\mathcal{P} =$ the set of linear polynomials and $\mathcal{N} = \{N_1, N_2\}$, where $N_1(v) = v(0)$ and $N_2(v) = v(1) \quad \forall v \in \mathcal{P}$. Then $(K, \mathcal{P}, \mathcal{N})$ is a finite element and the noda basis consists of $\phi_1(x) = 1 - x$ and $\phi_2(x) = x$.

In general, we can let $K = [a, b]$ and $\mathcal{P}_k =$ the set of all polynomials of degree less than or equal to k. Let $\mathcal{N}_k = \{N_0, N_1, N_2, \ldots, N_k\}$, where $N_i(v) = v(a + (b-a)i/k) \quad \forall v \in \mathcal{P}_k$ and $i = 0, 1, \ldots, k$. Then $(K, \mathcal{P}_k, \mathcal{N}_k)$ is a finite element. The verification of this uses Lemma 3.1.4.

Usually, condition (iii) of Definition 3.1.1 is the only one that requires much work, and the following simplifies its verification.

(3.1.4) Lemma. *Let $\mathcal{P}$ be a d-dimensional vector space and let $\{N_1, N_2, \ldots, N_d\}$ be a subset of the dual space $\mathcal{P}'$. Then the following two statements are equivalent.*

(a) *$\{N_1, N_2, \ldots, N_d\}$ is a basis for $\mathcal{P}'$.*
(b) *Given $v \in \mathcal{P}$ with $N_i v = 0$ for $i = 1, 2, \ldots, d$, then $v \equiv 0$.*

Proof. Let $\{\phi_1, \ldots, \phi_d\}$ be some basis for $\mathcal{P}$. $\{N_1, \ldots, N_d\}$ is a basis for $\mathcal{P}'$ iff given any L in $\mathcal{P}'$,

$$L = \alpha_1 N_1 + \ldots + \alpha_d N_d \tag{3.1.5}$$

(because $d = \dim \mathcal{P} = \dim \mathcal{P}'$). The equation (3.1.5) is equivalent to

$$y_i := L(\phi_i) = \alpha_1 N_1(\phi_i) + \ldots + \alpha_d N_d(\phi_i), \quad i = 1, \ldots, d.$$

Let $\mathbf{B} = \big(N_j(\phi_i)\big)$, $i, j = 1, \ldots, d$. Thus, (a) is equivalent to $\mathbf{B}\alpha = y$ *is always solvable*, which is the same as $\mathbf{B}$ being invertible.

Given any $v \in \mathcal{P}$, we can write $v = \beta_1 \phi_1 + \ldots + \beta_d \phi_d$. $N_i v = 0$ means that $\beta_1 N_i(\phi_1) + \ldots + \beta_d N_i(\phi_d) = 0$. Therefore, (b) is equivalent to

$$\begin{aligned} \beta_1 N_i(\phi_1) + \ldots + \beta_d N_i(\phi_d) = 0 \quad \text{for } i = 1, \ldots, d \\ \Longrightarrow \beta_1 = \ldots = \beta_d = 0. \end{aligned} \tag{3.1.6}$$

Let $\mathbf{C}= \big(N_i(\phi_j)\big)$, $i, j = 1, \ldots, d$. Then (b) is equivalent to $\mathbf{C}x = 0$ *only has trivial solutions*, which is the same as $\mathbf{C}$ being invertible. But $\mathbf{C} = \mathbf{B}^T$. Therefore, (a) is equivalent to (b). □

(3.1.7) *Remark.* Condition (iii) of Definition 3.1.1 is the same as (a) in Lemma 3.1.4, which can be verified by checking (b) in Lemma 3.1.4. For instance, in Example 3.1.3, $v \in \mathcal{P}_1$ means $v = a + bx$; $N_1(v) = N_2(v) = 0$ means $a = 0$ and $a + b = 0$. Hence, $a = b = 0$, i.e., $v \equiv 0$. More generally, if

$v \in \mathcal{P}_k$ and $0 = N_i(v) = v(a + (b-a)i/k)\ \forall i = 0, 1, \ldots, k$ then v vanishes identically by the fundamental theorem of algebra. Thus, $(K, \mathcal{P}_k, \mathcal{N}_k)$ is a finite element.

We will use the following terminology in subsequent sections.

(3.1.8) Definition. *We say that* $\mathcal{N}$ **determines** $\mathcal{P}$ *if* $\psi \in \mathcal{P}$ *with* $N(\psi) = 0 \quad \forall N \in \mathcal{N}$ *implies that* $\psi = 0$.

(3.1.9) *Remark.* We will often refer to the hyperplane $\{x : L(x) = 0\}$, where L is a non-degenerate linear function, simply as L.

(3.1.10) Lemma. *Let P be a polynomial of degree $d \geq 1$ that vanishes on a hyperplane L. Then we can write $P = LQ$, where Q is a polynomial of degree $(d-1)$.*

Proof. Make an affine change of coordinates such that $L(\hat{x}, x_n) = x_n$ and the hyperplane $L(\hat{x}, x_n) = 0$ is the $\hat{x}$-axis. Therefore, $P(\hat{x}, 0) \equiv 0$. Since degree$(P) = d$, we have

$$P(\hat{x}, x_n) = \sum_{j=0}^{d} \sum_{|\hat{i}| \leq d-j} c_{\hat{i}j} \hat{x}^{\hat{i}} x_n^j$$

where $\hat{x} = (x_1, \ldots, x_{n-1})$ and $\hat{i} = (i_1, \ldots, i_{n-1})$. Letting $x_n = 0$, we obtain $0 \equiv P(\hat{x}, 0) = \sum_{|\hat{i}| \leq d} c_{\hat{i}0} \hat{x}^{\hat{i}}$, which implies that $c_{\hat{i}0} = 0$ for $|\hat{i}| \leq d$. Therefore,

$$\begin{aligned} P(\hat{x}, x_n) &= \sum_{j=1}^{d} \sum_{|\hat{i}| \leq d-j} c_{\hat{i}j} \hat{x}^{\hat{i}} x_n^j \\ &= x_n \sum_{j=1}^{d} \sum_{|\hat{i}| \leq d-j} c_{\hat{i}j} \hat{x}^{\hat{i}} x_n^{j-1} \\ &= x_n Q \\ &= L Q, \end{aligned}$$

where degree $Q = d - 1$. □

3.2 Triangular Finite Elements

Let K be any triangle. Let $\mathcal{P}_k$ denote the set of all polynomials in two variables of degree $\leq k$. The following table gives the dimension of $\mathcal{P}_k$.

Table 3.1. Dimension of $\mathcal{P}_k$ in two dimensions

k	$\dim \mathcal{P}_k$
1	3
2	6
3	10
$\vdots$	$\vdots$
k	$\frac{1}{2}(k+1)(k+2)$

The Lagrange Element

(3.2.1) Example. (**k = 1**) Let $\mathcal{P} = \mathcal{P}_1$. Let $\mathcal{N}_1 = \{N_1, N_2, N_3\}$ ($\dim \mathcal{P}_1 = 3$) where $N_i(v) = v(z_i)$ and z_1, z_2, z_3 are the vertices of K. This element is depicted in Fig. 3.1.

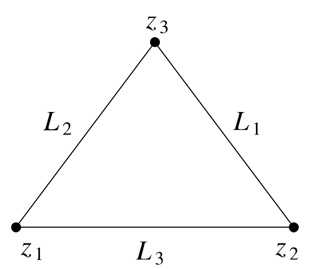

Fig. 3.1. linear Lagrange triangle

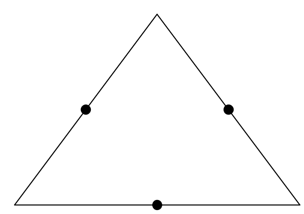

Fig. 3.2. Crouzeix-Raviart nonconforming linear triangle

Note that "•" indicates the nodal variable evaluation at the point where the dot is located.

We verify 3.1.1(iii) using 3.1.4(b), i.e., we prove that $\mathcal{N}_1$ determines $\mathcal{P}_1$. Let L_1, L_2 and L_3 be non-trivial linear functions that define the lines on which lie the edges of the triangle. Suppose that a polynomial $P \in \mathcal{P}$ vanishes at z_1, z_2 and z_3. Since $P|_{L_1}$ is a linear function of one variable that vanishes at two points, $P = 0$ on L_1. By Lemma 3.1.10 we can write $P = c\,L_1$, where c is a constant. But

$$0 = P(z_1) = c\,L_1(z_1) \quad \Longrightarrow \quad c = 0$$

(because $L_1(z_1) \neq 0$). Thus, $P \equiv 0$ and hence $\mathcal{N}_1$ determines $\mathcal{P}_1$. □

(3.2.2) *Remark.* The above choice for $\mathcal{N}$ is not unique. For example, we could have defined

$$N_i(v) = v(\text{midpoint of the } i^{th} \text{ edge}),$$

as shown in Fig. 3.2. By connecting the midpoints, we construct a triangle on which $P \in \mathcal{P}_1$ vanishes at the vertices. An argument similar to the one in Example 3.2.1 shows that $P \equiv 0$ and hence, $\mathcal{N}_1$ determines $\mathcal{P}_1$.

(3.2.3) Example. (**k = 2**) Let $\mathcal{P} = \mathcal{P}_2$. Let $\mathcal{N}_2 = \{N_1, N_2, \ldots, N_6\}$ ($\dim \mathcal{P}_2 = 6$) where

$$N_i(v) = \begin{cases} v(i^{th}\,\text{vertex}), & \text{i=1,2,3;} \\ v(\text{midpoint of the}\,(i-3)\ \text{edge}), & \\ \quad \text{(or any other point on the } i-3 \text{ edge)} & \text{i=4,5,6.} \end{cases}$$

This element is depicted in Fig. 3.3.

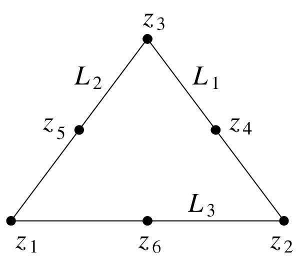

Fig. 3.3. quadratic Lagrange triangle

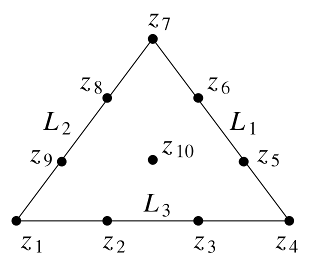

Fig. 3.4. cubic Lagrange triangle

We need to check that $\mathcal{N}_2$ determines $\mathcal{P}_2$. As before, let L_1, L_2 and L_3 be non-trivial linear functions that define the edges of the triangle. Suppose that the polynomial $P \in \mathcal{P}_2$ vanishes at $z_1, z_2, \ldots, z_6$. Since $P|_{L_1}$ is a quadratic function of one variable that vanishes at three points, $P = 0$ on L_1. By Lemma 3.1.10 we can write $P = L_1\, Q_1$ where $\deg Q_1 = (\deg P) - 1 = 2 - 1 = 1$. But P also vanishes on L_2. Therefore, $L_1\, Q_1|_{L_2} = 0$. Hence, on L_2, either $L_1 = 0$ or $Q_1 = 0$. But L_1 can equal zero only at one point of L_2 since we have a non-degenerate triangle. Therefore, $Q_1 = 0$ on L_2, except possibly at one point. By continuity, we have $Q_1 \equiv 0$ on L_2.

By Lemma 3.1.10, we can write $Q_1 = L_2\, Q_2$, where $\deg Q_2 = (\deg L_2) - 1 = 1 - 1 = 0$. Hence, Q_2 is a constant (say c), and we can write $P = c\, L_1\, L_2$. But $P(z_6) = 0$ and z_6 does not lie on either L_1 or L_2. Therefore,

$$0 = P(z_6) = c\, L_1(z_6)\, L_2(z_6) \quad \Longrightarrow \quad c = 0,$$

since $L_1(z_6) \neq 0$ and $L_2(z_6) \neq 0$. Thus, $P \equiv 0$. □

(3.2.4) Example. (**k=3**) Let $\mathcal{P} = \mathcal{P}_3$. Let $\mathcal{N}_3 = \{N_i : i = 1, 2, \ldots, 10\,(= \dim \mathcal{P}_3)\}$ where

$$N_i(v) = v(z_i), \quad i = 1, 2, \ldots, 9\ (z_i\ \text{distinct points on edges as in Fig. 3.4})$$

and

$$N_{10}(v) = v(\text{any interior point}).$$

We must show that $\mathcal{N}_3$ determines $\mathcal{P}_3$.

Let L_1, L_2 and L_3 be non-trivial linear functions that define the edges of the triangle. Suppose that $P \in \mathcal{P}_3$ vanishes at z_i for $i = 1, 2, \ldots, 10$. Applying Lemma 3.1.10 three times along with the fact that $P(z_i) = 0$ for $i = 1, 2, \ldots, 9$, we can write $P = c\, L_1\, L_2\, L_3$. But

$$0 = P(z_{10}) = c\, L_1(z_{10})\, L_2(z_{10})\, L_3(z_{10}) \quad \Longrightarrow \quad c = 0$$

since $L_i(z_{10}) \neq 0$ for $i = 1, 2, 3$. Thus, $P \equiv 0$. □

In general for $k \geq 1$, we let $\mathcal{P} = \mathcal{P}_k$. For $\mathcal{N}_k = \{N_i \;:\; i = 1, 2, \ldots, \frac{1}{2}(k+1)(k+2)\}$, we choose evaluation points at

$$\begin{aligned} &3\,\text{vertex nodes,} \\ &3(k-1)\,\text{distinct edge nodes and} \\ &\frac{1}{2}(k-2)(k-1)\ \text{ interior points.} \end{aligned} \tag{3.2.5}$$

(The interior points are chosen, by induction, to determine $\mathcal{P}_{k-3}$.) Note that these choices suffice since

$$\begin{aligned} 3 + 3(k-1) + \frac{1}{2}(k-2)(k-1) &= 3k + \frac{1}{2}(k^2 - 3k + 2) \\ &= \frac{1}{2}(k^2 + 3k + 2) \\ &= \frac{1}{2}(k+1)(k+2) \\ &= \dim \mathcal{P}_k. \end{aligned}$$

The evaluation points for $k = 4$ and $k = 5$ are depicted in Fig. 3.5.

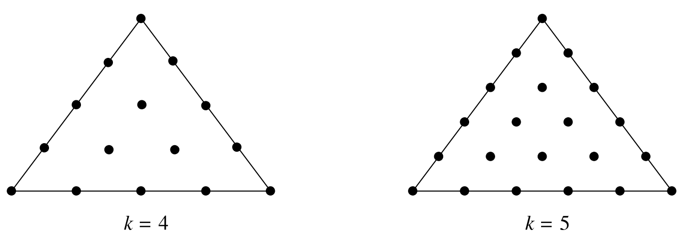

Fig. 3.5. quartic and quintic Lagrange triangles

To show that $\mathcal{N}_k$ determines $\mathcal{P}_k$, we suppose that $P \in \mathcal{P}_k$ vanishes at all the nodes. Let L_1, L_2 and L_3 be non-trivial linear functions that define the edges of the triangle. As before, we conclude from the vanishing of P at the edge and vertex nodes that $P = Q\, L_1\, L_2\, L_3$ where degree$(Q) \leq k - 3$; Q must vanish at all the interior points, since none of the L_i can be zero there. These points were chosen precisely to determine that $Q \equiv 0$.

The Hermite Element

(3.2.6) Example. (**k = 3 Cubic Hermite**) Let $\mathcal{P} = \mathcal{P}_3$. Let "•" denote evaluation at the point and "◯" denote evaluation of the gradient at the center of the circle. Note that the latter corresponds to two distinct nodal variables, but the particular representation of the gradient is not unique. We claim that $\mathcal{N} = \{N_1, N_2, \ldots, N_{10}\}$, as depicted in Fig. 3.6, determines $\mathcal{P}_3$ ($\dim \mathcal{P}_3 = 10$).

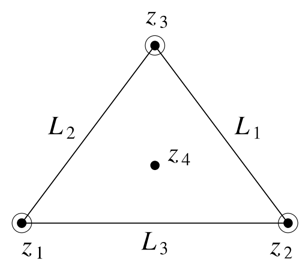

Fig. 3.6. cubic Hermite triangle

Let L_1, L_2 and L_3 again be non-trivial linear functions that define the edges of the triangle. Suppose that for a polynomial $P \in \mathcal{P}_3$, $N_i(P) = 0$ for $i = 1, 2, \ldots, 10$. Restricting P to L_1, we see that z_2 and z_3 are double roots of P since $P(z_2) = 0, P'(z_2) = 0$ and $P(z_3) = 0, P'(z_3) = 0$, where $'$ denotes differentiation along the straight line L_1. But the only third order polynomial in one variable with four roots is the zero polynomial, hence $P \equiv 0$ along L_1. Similarly, $P \equiv 0$ along L_2 and L_3. We can, therefore, write $P = c\, L_1\, L_2\, L_3$. But

$$0 = P(z_4) = c\, L_1(z_4)\, L_2(z_4)\, L_3(z_4) \quad \Longrightarrow \quad c = 0,$$

because $L_i(z_4) \neq 0$ for $i = 1, 2, 3$. □

(3.2.7) *Remark.* Using directional derivatives, there are various distinct ways to define a finite element using $\mathcal{P}_3$, two of which are shown in Fig. 3.7. Note that arrows represent directional derivatives along the indicated directions at the points. The "global" element to the left has the advantage of ease of computation of directional derivatives in the x or y directions throughout the larger region divided up into triangles. The "local" element to the right holds the advantage in that the nodal parameters of each triangle are invariant with respect to the triangle.

In the general Hermite case, we have

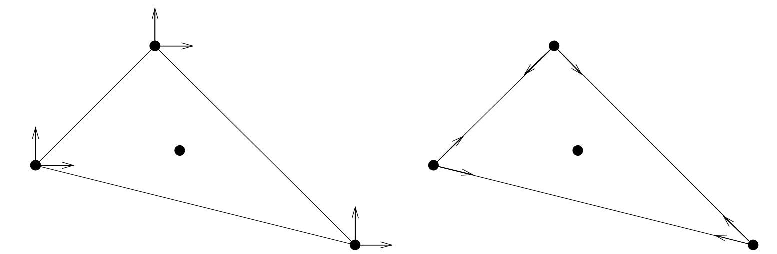

Fig. 3.7. Two different sets of nodal values for cubic Hermite elements.

$$
(3.2.9) \qquad \begin{cases} 3 \text{ vertex nodes} \\ 6 \text{ directional derivatives (2 for each gradient,} \\ \quad \text{evaluated at each of the 3 vertices)} \\ 3(k-3) \text{ edge nodes} \\ \dfrac{1}{2}(k-2)(k-1) \text{ interior nodes (as in the Lagrange case).} \end{cases}
$$

Note that these sum to $\frac{1}{2}(k+1)(k+2) = \dim \mathcal{P}_k$ as in the Lagrange case.

(3.2.8) Example. ($\mathbf{k = 4}$) We have ($\dim \mathcal{P}_4 = 15$). Then $\mathcal{N} = \{N_1, N_2, \ldots, N_{15}\}$, as depicted in Fig. 3.8, determines $\mathcal{P}_4$.

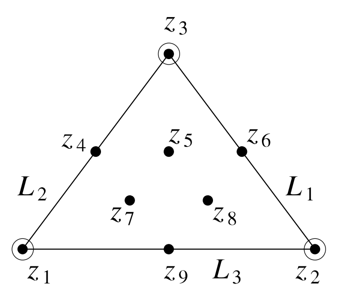

Fig. 3.8. quartic Hermite triangle

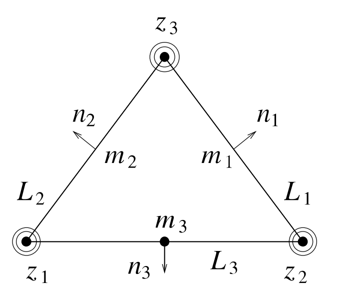

Fig. 3.9. quintic Argyris triangle

The Argyris Element

(3.2.10) Example. ($\mathbf{k = 5}$) Let $\mathcal{P} = \mathcal{P}_5$. Consider the $21 (= \dim \mathcal{P}_5)$ degrees of freedom shown in Fig. 3.9. As before, let $\bullet$ denote evaluation at the point and the inner circle denote evaluation of the gradient at the center. The outer circle denotes evaluation of the three second derivatives at the center. The arrows represent the evaluation of the normal derivatives at the three midpoints. We claim that $\mathcal{N} = \{N_1, N_2, \ldots, N_{21}\}$ determines $\mathcal{P}_5$.

Suppose that for some $P \in \mathcal{P}_5$, $N_i(P) = 0$ for $i = 1, 2, \ldots, 21$. Let L_i be as before in the Lagrange and Hermite cases. The restriction of P to L_1 is a fifth order polynomial in one variable with triple roots at z_2 and z_3. Hence,

P vanishes identically on L_1. Similarly, P vanishes on L_2 and L_3. Therefore, $P = Q\, L_1\, L_2\, L_3$, where $\deg Q = 2$. Observe that $(\partial_{L_1}\partial_{L_2} P)(z_3) = 0$, where ∂_{L_1} and ∂_{L_2} are the directional derivatives along L_1 and L_2 respectively. Therefore,

$$0 = (\partial_{L_1}\partial_{L_2} P)(z_3) = Q(z_3)\, L_3(z_3)\, \partial_{L_2} L_1\, \partial_{L_1} L_2,$$

since $\partial_{L_i} L_i \equiv 0$ & $L_i(z_3) = 0,\ i = 1, 2$. This implies $Q(z_3) = 0$ because $L_3(z_3) \neq 0$, $\partial_{L_2} L_1 \neq 0$ and $\partial_{L_1} L_2 \neq 0$. Similarly, $Q(z_1) = 0$ and $Q(z_2) = 0$. Also, since $L_1(m_1) = 0$, $\frac{\partial}{\partial n_1} P(m_1) = \big(Q \frac{\partial L_1}{\partial n_1}\, L_2\, L_3\big)(m_1)$. Therefore,

$$0 = \frac{\partial}{\partial n_1} P(m_1) \quad \Longrightarrow \quad Q(m_1) = 0$$

because $\frac{\partial L_1}{\partial n_1} \neq 0$, $L_2(m_1) \neq 0$ and $L_3(m_1) \neq 0$. Similarly, $Q(m_2) = 0$ and $Q(m_3) = 0$. So $Q \equiv 0$ by Example 3.2.3. □

We leave to the reader the verification of the following generalization of the Argyris element (exercise 3.x.12).

(3.2.11) Example. Note that $\dim \mathcal{P}_7 = 36$. The nodal variables depicted in Fig. 3.10 determine $\mathcal{P}_7$.

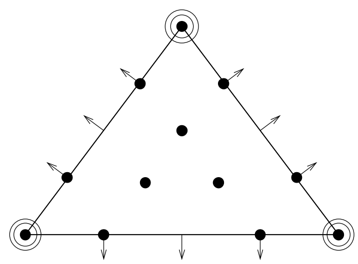

Fig. 3.10. seventh-degree Argyris triangle

3.3 The Interpolant

Now that we have examined a number of finite elements, we wish to piece them together to create subspaces of Sobolev spaces. We begin by defining the (local) interpolant.

(3.3.1) Definition. *Given a finite element* $(K, \mathcal{P}, \mathcal{N})$ *, let the set* $\{\phi_i : 1 \leq i \leq k\} \subseteq \mathcal{P}$ *be the basis dual to* $\mathcal{N}$*. If* v *is a function for which all* $N_i \in \mathcal{N}$*,* $i = 1, \ldots, k$*, are defined, then we define the* **local interpolant** *by*

$$\mathcal{I}_K v := \sum_{i=1}^{k} N_i(v)\, \phi_i. \tag{3.3.2}$$

(3.3.3) Example. Let K be the triangle depicted in Fig. 3.11, $\mathcal{P} = \mathcal{P}_1$, $\mathcal{N} = \{N_1, N_2, N_3\}$ as in Example 3.2.1, and $f = e^{xy}$. We want to find $\mathcal{I}_K f$.

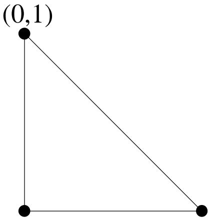

Fig. 3.11. coordinates for linear interpolant

By definition, $\mathcal{I}_K f = N_1(f)\,\phi_1 + N_2(f)\,\phi_2 + N_3(f)\,\phi_3$. We must therefore determine ϕ_1, ϕ_2 and ϕ_3. The line L_1 is given by $y = 1 - x$. We can write $\phi_1 = c\,L_1 = c(1 - x - y)$. But $N_1\,\phi_1 = 1$ implies that $c = \phi_1(z_1) = 1$, hence $\phi_1 = 1 - x - y$. Similarly, $\phi_2 = L_2(x,y)/L_2(z_2) = x$ and $\phi_3 = L_3(x,y)/L_3(z_3) = y$. Therefore,

$$\begin{aligned}\mathcal{I}_K f &= N_1(f)\,(1 - x - y) + N_2(f)\,x + N_3(f)\,y \\ &= 1 - x - y + x + y && (\text{since } f = e^{xy}) \\ &= 1.\end{aligned}$$

Properties of the interpolant follow. □

(3.3.4) Proposition. $\mathcal{I}_K$ *is linear.*

Proof. See exercise 3.x.2. □

(3.3.5) Proposition. $N_i\big(\mathcal{I}_K(f)\big) = N_i(f) \quad \forall 1 \le i \le d.$

Proof. We have

$$\begin{aligned}N_i\big(\mathcal{I}_K(f)\big) &= N_i\left(\sum_{j=1}^{k} N_j(f)\,\phi_j\right) && \big(\text{definition of } \mathcal{I}_K(f)\big) \\ &= \sum_{j=1}^{k} N_j(f)\,N_i(\phi_j) && \big(\text{linearity of } N_i\big) \\ &= N_i(f) && \big(\{\phi_j\} \text{ dual to } \{N_j\}\big).\end{aligned}$$

□

(3.3.6) *Remark.* Proposition 3.3.5 has the interpretation that $\mathcal{I}_K(f)$ is the unique shape function that has the same nodal values as f.

(3.3.7) Proposition. *$\mathcal{I}_K(f) = f$ for $f \in \mathcal{P}$. In particular, $\mathcal{I}_K$ is idempotent, i.e., $\mathcal{I}_K^2 = \mathcal{I}_K$.*

Proof. From (3.3.5),

$$N_i(f - \mathcal{I}_K(f)) = 0 \quad \forall i$$

which implies the first assertion. The second is a consequence of the first:

$$\mathcal{I}_K^2 f = \mathcal{I}_K(\mathcal{I}_K f) = \mathcal{I}_K f,$$

since $\mathcal{I}_K f \in \mathcal{P}$. □

We now piece together the elements.

(3.3.8) Definition. *A* **subdivision** *of a domain Ω is a finite collection of element domains $\{K_i\}$ such that*

(1) $\operatorname{int} K_i \cap \operatorname{int} K_j = \emptyset$ *if $i \neq j$ and*
(2) $\bigcup K_i = \overline{\Omega}$.

(3.3.9) Definition. *Suppose Ω is a domain with a subdivision $\mathcal{T}$. Assume each element domain, K, in the subdivision is equipped with some type of shape functions, $\mathcal{P}$, and nodal variables, $\mathcal{N}$, such that $(K, \mathcal{P}, \mathcal{N})$ forms a finite element. Let m be the order of the highest partial derivatives involved in the nodal variables. For $f \in C^m(\overline{\Omega})$, the* **global interpolant** *is defined by*

$$\mathcal{I}_{\mathcal{T}} f|_{K_i} = \mathcal{I}_{K_i} f \tag{3.3.10}$$

for all $K_i \in \mathcal{T}$.

Without further assumptions on a subdivision, no continuity properties can be asserted for the global interpolant. We now describe conditions that yield such continuity. Only the two-dimensional case using triangular elements is considered in detail here; analogous definitions and results can be formulated for higher dimensions and other subdivisions.

(3.3.11) Definition. *A* **triangulation** *of a polygonal domain Ω is a subdivision consisting of triangles having the property that*

(3) *no vertex of any triangle lies in the interior of an edge of another triangle.*

(3.3.12) Example. The figure on the left of Fig. 3.12 shows a triangulation of the given domain. The figure on the right is *not* a triangulation.

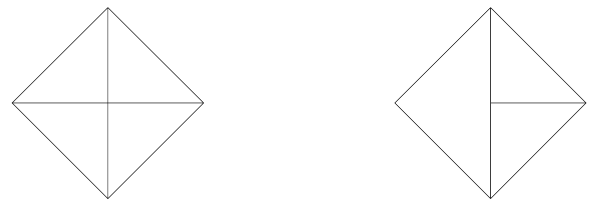

Fig. 3.12. Two subdivisions: the one on the left is a triangulation and the one on the right is not.

(3.3.13) Example. Let Ω be the square depicted in Fig. 3.13. The triangulation $\mathcal{T}$ consists of the two triangles T_1 and T_2, as indicated. The finite element on each triangle is the Lagrange element in Example 3.2.1. The dual basis on T_1 is $\{1-x-y, x, y\}$ (calculated in Example 3.3.3) and the dual basis on T_2 is (cf. exercise 3.x.3) $\{1-x, 1-y, x+y-1\}$. Let $f = \sin\big(\pi(x+y)/2\big)$. Then

$$\mathcal{I}_\mathcal{T} f = \begin{cases} x+y & \text{on } T_1 \\ 2-x-y & \text{on } T_2. \end{cases}$$

(3.3.14) *Remark.* For approximating the Dirichlet problem with zero boundary conditions, we use a finite-dimensional space of piecewise polynomial functions satisfying the boundary conditions given by

$$V_\mathcal{T} = \{\mathcal{I}_\mathcal{T} f : f \in C^m(\overline{\Omega}), f|_{\partial\Omega} = 0\}$$

on each triangulation $\mathcal{T}$. This will be discussed further in Chapter 5.

(3.3.15) Definition. *We say that an interpolant has* **continuity order** r *(in short, that it is "C^r") if $\mathcal{I}_\mathcal{T} f \in C^r$ for all $f \in C^m(\overline{\Omega})$. The space, $V_\mathcal{T} = \{\mathcal{I}_\mathcal{T} f \;:\; f \in C^m\}$, is said to be a "$C^r$" finite element space.*

(3.3.16) *Remark.* A finite element (or collection of elements) that can be used to form a C^r space as above is often called a "C^r element." Not all choices of nodes will always lead to C^r continuity, however. Some sort of regularity must be imposed. For the elements studied so far, the essential point is that they be placed in a coordinate-free way that is symmetric with respect to the midpoint of the edge.

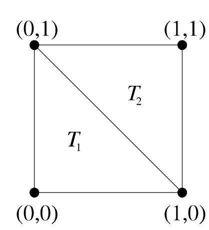

Fig. 3.13. simple triangulation consisting of two triangles

(3.3.17) Proposition. *The Lagrange and Hermite elements are both C^0 elements, and the Argyris element is C^1. More precisely, given a triangulation, $\mathcal{T}$, of Ω, it is possible to choose edge nodes for the corresponding elements $(K,\mathcal{P},\mathcal{N})$, $K \in \mathcal{T}$, such that the global interpolant satisfies $\mathcal{I}_{\mathcal{T}} f \in C^r$ ($r = 0$ for Lagrange and Hermite, and $r = 1$ for Argyris) for $f \in C^m$ ($m = 0$ for Lagrange, $m = 1$ for Hermite and $m = 2$ for Argyris). In particular, it is sufficient for each edge $\overline{\mathbf{x}\,\mathbf{x}'}$ to have nodes $\xi_i(\mathbf{x}' - \mathbf{x}) + \mathbf{x}$, where $\{\xi_i \,:\, i = 1, \ldots, k-1-2m\}$ is fixed and symmetric around $\xi = 1/2$. Moreover, under these hypotheses, $\mathcal{I}_{\mathcal{T}} f \in W^{r+1}_\infty$.*

Proof. It is sufficient to show that the stated continuity holds across each edge. Let T_i, $i = 1, 2$, denote two triangles sharing an edge, e. Since we assumed that the edge nodes were chosen symmetrically and in a coordinate-free way, we know that the edge nodes on e for the elements on both T_1 and T_2 are at the same location in space. Let $w := \mathcal{I}_{T_1} f - \mathcal{I}_{T_2} f$, where we view both polynomials, $\mathcal{I}_{T_i} f$ to be defined everywhere by extension outside T_i as polynomials. Then w is a polynomial of degree k and its restriction to the edge e has one-dimensional Lagrange, Hermite or Argyris nodes equal to zero. Thus, $w|_e$ must vanish. Hence, the interpolant is continuous across each edge.

Lipschitz continuity of $\mathcal{I}_{\mathcal{T}} f$ follows by showing that it has weak derivatives of order $r + 1$ given by

$$\big(D^\alpha_{(w)} \mathcal{I}_{\mathcal{T}} f\big)|_T = D^\alpha \mathcal{I}_T f \quad \forall T \in \mathcal{T},\ |\alpha| \le r+1.$$

The latter is certainly in L^∞. The verification that this is the weak derivative follows from

$$\begin{aligned}\int_\Omega (D^\alpha \phi)\,(\mathcal{I}_{\mathcal{T}} f)\;dx &= \sum_{T\in\mathcal{T}} \int_T (D^\alpha \phi)\,(\mathcal{I}_T f)\;dx \\ &= \sum_{T\in\mathcal{T}} (-1)^{|\alpha|} \int_T \phi\,(D^\alpha \mathcal{I}_T f)\;dx \\ &=(-1)^{|\alpha|} \int_\Omega \phi \sum_{T\in\mathcal{T}} \chi_T\,(D^\alpha \mathcal{I}_T f)\;dx,\end{aligned}$$

where χ_T denotes the characteristic function of T. The second equality holds because all boundary terms cancel due to the continuity properties of the interpolant. □

3.4 Equivalence of Elements

In the application of the global interpolant, it is essential that we find a uniform bound (independent of $T \in \mathcal{T}$) for the norm of the local interpolation operator $\mathcal{I}_T$. Therefore, we want to compare the local interpolation operators on different elements. The following notions of equivalence are useful for this purpose (cf. Ciarlet & Raviart 1972a).

(3.4.1) Definition. *Let* $(K,\mathcal{P},\mathcal{N})$ *be a finite element and let* $F(x) = \mathbf{A}x + \mathbf{b}$ ($\mathbf{A}$ *nonsingular) be an affine map. The finite element* $(\widehat{K},\widehat{\mathcal{P}},\widehat{\mathcal{N}})$ *is* **affine equivalent** *to* $(K,\mathcal{P},\mathcal{N})$ *if*

(i) $F(K) = \hat{K}$
(ii) $F^*\hat{\mathcal{P}} = \mathcal{P}$ *and*
(iii) $F_*\mathcal{N} = \hat{\mathcal{N}}$.

We write $(K,\mathcal{P},\mathcal{N}) \underset{F}{\simeq} (\widehat{K},\widehat{\mathcal{P}},\widehat{\mathcal{N}})$ if they are affine equivalent.

(3.4.2) *Remark.* Recall that the *pull-back* F^* is defined by $F^*(\hat{f}) := \hat{f} \circ F$ and the *push-forward* F_* is defined by $(F_*N)(\hat{f}) := N\big(F^*(\hat{f})\big)$.

(3.4.3) Proposition. *Affine equivalence is an equivalence relation.*

Proof. See exercise 3.x.4. □

(3.4.4) Examples.

(i) Let K be any triangle, $\mathcal{P} = \mathcal{P}_1$, $\mathcal{N} = \{$evaluation at vertices of $K\}$. All such elements $(K,\mathcal{P},\mathcal{N})$ are affine equivalent.
(ii) Let K be any triangle, $\mathcal{P} = \mathcal{P}_2$, $\mathcal{N} = \{$evaluation at vertices and edge midpoints$\}$. All such elements are affine equivalent.
(iii) Let $\mathcal{P} = \mathcal{P}_2$. In Fig. 3.14, $(T_1,\mathcal{P},\mathcal{N}_1)$ and $(T_2,\mathcal{P},\mathcal{N}_2)$ are *not* affine equivalent, but the finite elements $(T_1,\mathcal{P},\mathcal{N}_1)$ and $(T_3,\mathcal{P},\mathcal{N}_3)$ are affine equivalent.

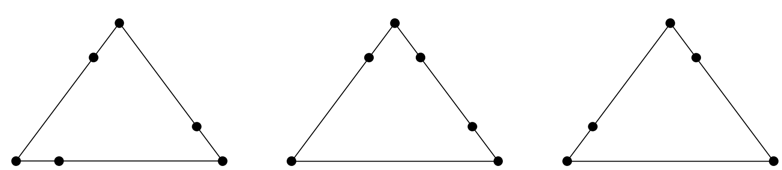

Fig. 3.14. inequivalent quadratic elements: noda placement incompatibility

(iv) Let $\mathcal{P} = \mathcal{P}_3$. The elements $(T_1,\mathcal{P},\mathcal{N}_1)$ and $(T_2,\mathcal{P},\mathcal{N}_2)$ depicted in Fig. 3.15 are *not* affine equivalent since the directional derivatives differ.
(v) Let $\mathcal{P} = \mathcal{P}_3$. Then the elements $(T_1,\mathcal{P},\mathcal{N}_1)$ and $(T_2,\mathcal{P},\mathcal{N}_2)$ depicted in Fig. 3.16 are *not* affine equivalent since the strength of the directional derivatives (indicated by the length of the arrows) differ.

(3.4.5) Proposition. *There exist nodal placements such that all Lagrange elements of a given degree are affine equivalent.*

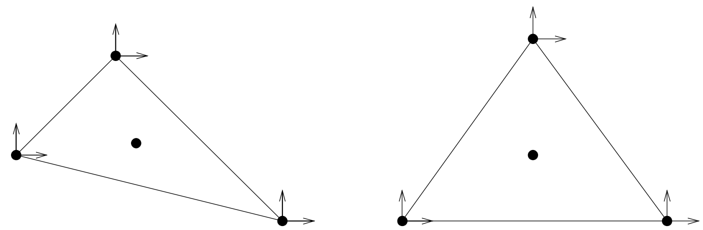

Fig. 3.15. inequivalent cubic Hermite elements: direction incompatibility

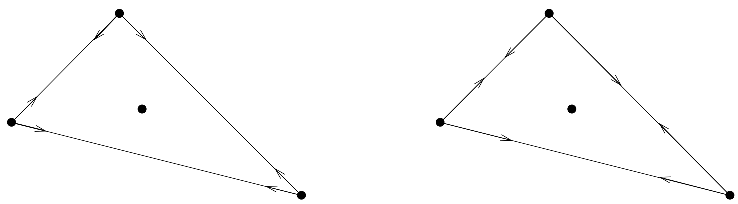

Fig. 3.16. inequivalent cubic Hermite elements: derivative strength incompatibility

Proof. We pick nodes using *barycentric coordinates*, (b_1, b_2, b_3), for each triangle. The i-th barycentric coordinate of a point (x, y) can be defined simply as the value of the i-th linear Lagrange basis function at that point $(b_i(x, y) := \phi_i(x, y))$. Thus, each barycentric coordinate is naturally associated with a given vertex; it is equal to the proportional distance of the point from the opposite edge. Note that the barycentric coordinates sum to one (since this yields the interpolant of the constant, 1). Thus, the mapping $(x, y) \to \mathbf{b}(x, y)$ maps the triangle (invertibly) to a subset of $\{\mathbf{b} \in [0,1]^3 \ : \ b_1 + b_2 + b_3 = 1\}$.

For degree k Lagrange elements, pick nodes at the points whose barycentric coordinates are

$$\left(\frac{i}{k}, \frac{j}{k}, \frac{l}{k}\right) \quad \text{where} \quad 0 \leq i, j, l \leq k \quad \text{and} \quad i + j + l = k.$$

□

(3.4.6) Definition. *The finite elements* $(K, \mathcal{P}, \mathcal{N})$ *and* $(K, \mathcal{P}, \widetilde{\mathcal{N}})$ *are* **interpolation equivalent** *if*

$$\mathcal{I}_{\mathcal{N}} f = \mathcal{I}_{\widetilde{\mathcal{N}}} f \quad \forall f \textit{ sufficiently smooth},$$

where $\mathcal{I}_{\mathcal{N}}$ *(resp.* $\mathcal{I}_{\widetilde{\mathcal{N}}}$*) is defined by the right-hand side of* (3.3.2) *with* $N_i \in \mathcal{N}$ *(resp.* $N_i \in \widetilde{\mathcal{N}}$*). We write* $(K, \mathcal{P}, \mathcal{N}) \underset{\mathcal{I}}{\simeq} (K, \mathcal{P}, \widetilde{\mathcal{N}})$.

(3.4.7) Proposition. *Suppose* $(K,\mathcal{P},\mathcal{N})$ *and* $(K,\mathcal{P},\widetilde{\mathcal{N}})$ *are finite elements. Every nodal variable in* $\mathcal{N}$ *is a linear combination of nodal variables in* $\widetilde{\mathcal{N}}$ *(when viewed as a subset of* $C^m(K)'$*) if and only if* $(K,\mathcal{P},\mathcal{N}) \underset{\mathcal{I}}{\simeq} (K,\mathcal{P},\widetilde{\mathcal{N}})$.

Proof. (*only if*) We must show that $\mathcal{I}_{\mathcal{N}} f = \mathcal{I}_{\widetilde{\mathcal{N}}} f \quad \forall f \in C^m(K)$. For $N_i \in \mathcal{N}$, we can write $N_i = \sum_{j=1}^k c_j \tilde{N}_j$ since every nodal variable in $\mathcal{N}$ is a linear combination of nodal variables in $\widetilde{\mathcal{N}}$. Therefore,

$$
\begin{aligned}
N_i(\mathcal{I}_{\widetilde{\mathcal{N}}} f) &= \left(\sum_{j=1}^k c_j \, \tilde{N}_j \right) (\mathcal{I}_{\widetilde{\mathcal{N}}} f) \\
&= \sum_{j=1}^k c_j \, \tilde{N}_j (\mathcal{I}_{\widetilde{\mathcal{N}}} f) \\
&= \sum_{j=1}^k c_j \, \tilde{N}_j (f) \\
&= N_i(f).
\end{aligned}
$$

The converse is left to the reader in exercise 3.x.26. □

(3.4.8) Example. The Hermite elements in Fig. 3.7 (and 3.15–16) are interpolation equivalent (exercise 3.x.29).

(3.4.9) Definition. *If* $(K,\mathcal{P},\mathcal{N})$ *is a finite element that is affine equivalent to* $(\widehat{K},\widehat{\mathcal{P}},\widehat{\mathcal{N}})$ *and* $(\widehat{K},\widehat{\mathcal{P}},\widehat{\mathcal{N}})$ *is interpolation equivalent to* $(\widetilde{K},\widetilde{\mathcal{P}},\widetilde{\mathcal{N}})$*, then we say that* $(K,\mathcal{P},\mathcal{N})$ *is* **affine-interpolation equivalent** *to* $(\widetilde{K},\widetilde{\mathcal{P}},\widetilde{\mathcal{N}})$.

(3.4.10) Example.

(i) All affine equivalent elements (e.g., Lagrange elements with appropriate choices for the edge and interior nodes as described in Proposition 3.4.5) are affine-interpolation equivalent.
(ii) The Hermite elements with appropriate choices for the edge and interior nodes are affine-interpolation equivalent.
(iii) The Argyris elements are *not* affine-interpolation equivalent (Ciarlet 1978).

The following is an immediate consequence of the definitions.

(3.4.11) Proposition. *If* $(K,\mathcal{P},\mathcal{N})$ *is affine-interpolation equivalent to* $(\widetilde{K},\widetilde{\mathcal{P}},\widetilde{\mathcal{N}})$ *then* $\mathcal{I} \circ F^* = F^* \circ \widetilde{\mathcal{I}}$ *where* F *is the affine mapping* $K \to \widetilde{K}$.

3.5 Rectangular Elements

In this section we consider finite elements defined on rectangles. Let $\mathcal{Q}_k = \{\sum_j c_j\, p_j(x)\, q_j(y) : p_j, q_j$ polynomials of degree $\le k\}$. One can show that

$$\dim \mathcal{Q}_k = (\dim \mathcal{P}_k^1)^2, \tag{3.5.1}$$

where $\mathcal{P}_k^1$ denotes the space of polynomials of degree less than or equal to k in one variable (cf. exercise 3.x.6).

Tensor Product Elements

(3.5.2) Example. (**k = 1**) Let K be any rectangle, $\mathcal{P} = \mathcal{Q}_1$, and $\mathcal{N}$ as depicted in Fig. 3.17.

Suppose that the polynomial $P \in \mathcal{Q}_1$ vanishes at z_1, z_2, z_3 and z_4. The restriction of P to any side of the rectangle is a first-order polynomial of one variable. Therefore, we can write $P = c\, L_1 L_2$ for some constant c. But

$$0 = P(z_4) = c\, L_1(z_4)\, L_2(z_4) \quad \Longrightarrow \quad c = 0,$$

since $L_1(z_4) \neq 0$ and $L_2(z_4) \neq 0$. Thus, $P \equiv 0$. □

(3.5.3) Example. (**k = 2**) Let K be any rectangle, $\mathcal{P} = \mathcal{Q}_2$, and $\mathcal{N}$ as depicted in Fig. 3.18. Suppose that a polynomial $P \in \mathcal{Q}_2$ vanishes at z_i, for $i = 1, \ldots, 9$. Then we can write $P = c\, L_1 L_2 L_3 L_4$ for some constant c. But

$$0 = P(z_9) = c\, L_1(z_9)\, L_2(z_9)\, L_3(z_9)\, L_4(z_9) \quad \Longrightarrow \quad c = 0,$$

since $L_i(z_9) \neq 0$ for $i = 1, 2, 3, 4$. □

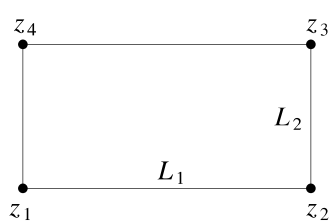

Fig. 3.17. bilinear Lagrange rectangle

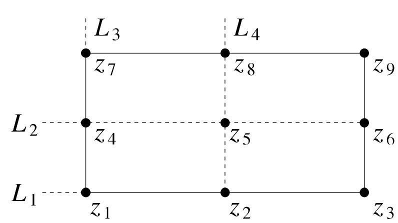

Fig. 3.18. biquadratic Lagrange rectangle

(3.5.4) Example. (**arbitrary k**) Let K be any rectangle, $\mathcal{P} = \mathcal{Q}_k$, and $\mathcal{N}$ denote point evaluations at $\{(t_i, t_j) \ :\ i, j = 0, 1, \ldots k\}$ where $\{0 = t_0 < t_1 < \ldots < t_k = 1\}$. (The case $k = 3$ is depicted in Fig. 3.19.) Then $(K, \mathcal{P}, \mathcal{N})$ is a finite element (cf. exercise 3.x.7).

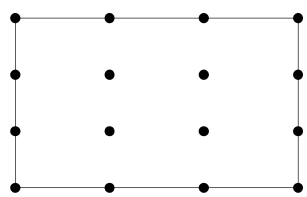

Fig. 3.19. bicubic Lagrange rectangle

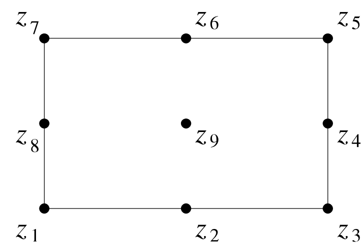

Fig. 3.20. notation for Lemma 3.5.6

The Serendipity Element

(3.5.5) Example. (**Quadratic Case**) To define the shape functions for this case, we need the following lemma (see Fig. 3.20 for the notation).

(3.5.6) Lemma. *There exist constants* $c_1, \ldots, c_8$ *such that*

$$\phi(z_9) = \sum_{i=1}^{8} c_i\, \phi(z_i) \quad \textit{for } \phi \in \mathcal{P}_2.$$

Proof. Note that evaluation at $z_1, \ldots, z_6$ forms a nodal basis for $\mathcal{P}_2$. Let $\{\phi_1, \ldots, \phi_6\}$ be the dual basis of $\mathcal{P}_2$, i.e., $N_i\phi_j = \delta_{ij}$ for $i, j = 1, \ldots, 6$. If $\phi \in \mathcal{P}_2$, then

$$\phi = N_1(\phi)\,\phi_1 + N_2(\phi)\,\phi_2 + \ldots + N_6(\phi)\,\phi_6.$$

Therefore,

$$\begin{aligned}\phi(z_9) &= \phi(z_1)\,\phi_1(z_9) + \phi(z_2)\,\phi_2(z_9) + \ldots + \phi(z_6)\,\phi_6(z_9)\\ &= c_1\,\phi(z_1) + c_2\,\phi(z_2) + \ldots + c_8\,\phi(z_8)\end{aligned}$$

(let $c_7 = c_8 = 0$). □

Let K be any rectangle, $\mathcal{P} = \{\phi \in \mathcal{Q}_2 : \sum_{i=1}^{8} c_i\,\phi(z_i) - \phi(z_9) = 0\}$, and $\mathcal{N}$ as depicted in Fig. 3.21. Then $(K, \mathcal{P}, \mathcal{N})$ is a finite element because if $\phi \in \mathcal{P}$ vanishes at $z_1, \ldots, z_8$ we can write $\phi = c\,L_1L_2L_3L_4$ for some constant c. But

$$0 = \sum_{i=1}^{8} c_i\,\phi(z_i) = \phi(z_9) \quad \Longrightarrow \quad c = 0$$

(since $L_i(z_9) \neq 0$, $i = 1, \ldots, 4$). Therefore, $\phi \equiv 0$. □

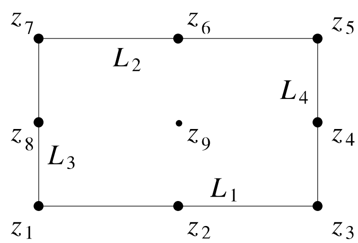

Fig. 3.21. quadratic serendipity element

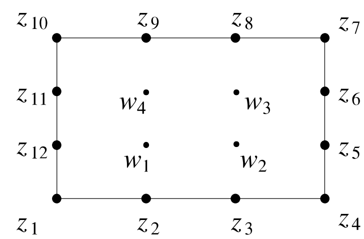

Fig. 3.22. cubic serendipity element

(3.5.7) Example. (**Cubic Case**) Let K be any rectangle. There exist constants c^i_j such that for $\phi \in \mathcal{P}_3$, $\phi(w_i) = \sum_{j=1}^{12} c^i_j \, \phi(z_j)$ $i = 1,2,3,4$ (for w_i and z_j as depicted in Fig. 3.22), then let $\mathcal{P} = \{\phi \in \mathcal{Q}_3 : \phi(w_i) - \sum_{j=1}^{12} c^i_j \, \phi(z_j) = 0$ for $i = 1,2,3,4\}$ and $\mathcal{N}$ as depicted. Then $(K, \mathcal{P}, \mathcal{N})$ is a finite element (cf. exercise 3.x.8).

(3.5.8) *Remark.* The notion of a C^r rectangular element can be defined similarly to Definition 3.3.15. Following the proof of Proposition 3.3.17, we can see that all the rectangular elements defined in this section are C^0. An example of a C^1 rectangular element is in exercise 3.x.16.

(3.5.9) *Remark.* The space, $\mathcal{P}$, of shape functions for serendipity elements is not uniquely defined by the choice of nodal variables. Another way to choose them is described in Sect. 4.6.

3.6 Higher-dimensional Elements

Higher-dimensional elements can be constructed inductively just the way we constructed two-dimensional elements using properties of one-dimensional elements as building blocks. As an illustration, we describe tetrahedral elements in three dimensions. The Lagrange elements can be defined as before, inductively in the degree, k, as follows.

We pick nodal variables at the vertices (of which there are four), at $k-1$ points on the interior of each edge (there are six edges) and at $(k-2)(k-1)/2$ points in the interior of each face (again four of these). The face points, which exist only for $k \geq 3$, should be chosen so as to determine polynomials in two variables (in the plane of the face) of degree $k-3$. For $k \geq 4$, we also pick points in the interior of the tetrahedron so as to determine polynomials in three variables of degree $k-4$. The existence of the latter will be demonstrated by induction on k, as in the two-dimensional case.

Suppose these nodal values vanish for $v \in \mathcal{P}_k$. The restriction of v to each face, F_i, of the tetrahedron is a polynomial in two variables (the coordinates for the plane of F_i), and the nodal variables have been chosen to determine this restriction. Thus, $v|_{F_i}$ is identically zero. Let L_i denote a nontrivial linear function vanishing on F_i. By applying (3.1.10) four times,

$$v = L_1 L_2 L_3 L_4 R$$

where the remainder is a polynomial of degree $k-4$. For $k \le 3$ this implies that $R = 0$, so that $v = 0$. In the general case, we use the interior nodes to determine that $R = 0$. It simply remains to count the number of nodes and check that it equals $\dim \mathcal{P}_k$.

We have enumerated

$$C(k) := 4 + 6(k-1) + 2(k-2)(k-1) + \dim \mathcal{P}_{k-4} \tag{3.6.1}$$

nodes above. The dimension of $\mathcal{P}_k$ can be computed as follows. We can decompose an arbitrary polynomial, P, of degree k in three variables uniquely as

$$P(x,y,z) = p(x,y) + zq(x,y,z) \tag{3.6.2}$$

where the degree of p is k and the degree of q is $k-1$. Simply let $p(x,y) := P(x,y,0)$ and apply (3.1.10) to $P - p$ with $L(x,y,z) = z$. Therefore,

$$\dim \mathcal{P}_k = (k+1)(k+2)/2 + \dim \mathcal{P}_{k-1} = \sum_{j=0}^{k} (j+1)(j+2)/2, \tag{3.6.3}$$

where the second equality follows from the first by induction. The first few of these are given in the following table, and it is easily checked that they agree with (3.6.1).

Table 3.2. dimension of polynomials of degree k, $\mathcal{P}_k$, in three dimensions

k	dim $\mathcal{P}_k$
1	4
2	10
3	20
4	35

Since (3.6.3) implies $\dim \mathcal{P}_k$ is a cubic polynomial in k (with leading coefficient $1/6$), we conclude that $C(k)$ is also a cubic polynomial in k. Since these cubics agree for $k = 1, 2, 3, 4$, they must be identical.

The above arguments also show that the nodes can be arranged so as to insure that the Lagrange elements are C^0. As in the proof of Proposition 3.3.17, it suffices to see that the restrictions of the global interpolant to neighboring tetrahedra agree on the common face. This is possible because of our choice of facial nodes to determine polynomials in two variables on that face. One must again choose the facial nodes in a symmetric and coordinate free way. In particular, it is sufficient to let the nodes be located at points whose barycentric coordinates (see (3.4.5)) B on each face satisfy

$$(b_1, b_2, b_3) \in B \Longrightarrow (b_{\sigma(1)}, b_{\sigma(2)}, b_{\sigma(3)}) \in B$$

for any permutation σ of the indices.

(3.6.4) *Remark.* The notion of a C^r tetrahedral element can be defined similarly to Definition 3.3.15. Following the proof of Proposition 3.3.17, we see that the tetrahedral elements defined in this section are all C^0.

3.7 Exotic Elements

All the elements $(K, \mathcal{P}, \mathcal{N})$ studied so far have shape functions consisting of polynomials. However, this is not at all necessary. We consider some of the possibilities briefly here. We restrict our discussion to the class of *macro*-finite-elements, for which the shape functions, $\mathcal{P}$, are themselves piecewise polynomials. Other types of shape functions have been proposed, e.g., rational functions (Wachspress 1975).

Let K denote a triangle, and let it be divided into four subtriangles by connecting edge midpoints as shown in Fig. 3.23. Define $\mathcal{P}$ to be the set of continuous piecewise linear functions on this subtriangulation. If $\mathcal{N}$ consists of point-evaluations at the vertices and edge midpoints of K, we clearly have a well defined C^0 finite element.

A more complex element is that of Clough and Tocher (Ciarlet 1978). Let K denote a triangle, and let it be divided into three subtriangles as shown in Fig. 3.24. Let $\mathcal{P}$ be the set of C^1 piecewise cubic functions on this subtriangulation. Let $\mathcal{N}$ consist of point- and gradient-evaluations at the vertices and normal-derivative-evaluations at the edge midpoints of K. Then $(K, \mathcal{P}, \mathcal{N})$ is a well defined, C^1 finite element (Ciarlet 1978).

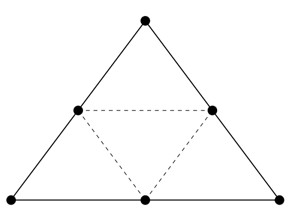

Fig. 3.23. macro-piecewise-linear triangle

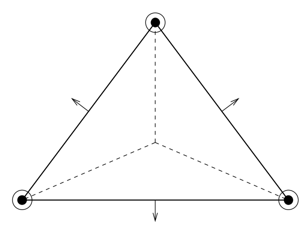

Fig. 3.24. Clough-Tocher C^1 macro-piecewise-cubic triangle

3.x Exercises

3.x.1 Let m and k be nonnegative integers, and let P be a polynomial in one variable of degree $2m+k+1$. Suppose that $P^{(j)}(a) = 0$ for $a = 0, 1$ and $j = 0, \ldots, m$, and further that $P(\xi_j) = 0$ for $0 < \xi_1 < \ldots < \xi_k < 1$. Prove that $P \equiv 0$.

3.x.2 Prove that the local interpolant is linear (cf. Proposition 3.3.4).

3.x.3 Find the dual basis for triangle T_2 in Example 3.3.13.

3.x.4 Show that affine equivalence is an equivalence relation (cf. Proposition 3.4.3).

3.x.5 Show that interpolation equivalence is an equivalence relation.

3.x.6 Show that $\dim \mathcal{Q}_k = (\dim \mathcal{P}_k^1)^2$, where $\mathcal{P}_k^1 = \{$polynomials in one variable of degree less than or equal to $k\}$ and $\{x^i y^j : i, j = 0, \ldots, k\}$ is a basis of $\mathcal{Q}_k$.

3.x.7 Prove that $(K, \mathcal{P}, \mathcal{N})$ in Example 3.5.4 is a finite element.

3.x.8 Prove that $(K, \mathcal{P}, \mathcal{N})$ in Example 3.5.7 is a finite element.

3.x.9 Construct nodal basis functions for $K =$ the rectangle with vertices $(-1, 0)$, $(1, 0)$, $(1, 1)$ and $(-1, 1)$, $\mathcal{P} = \mathcal{Q}_1$, and $\mathcal{N} =$ evaluation at the vertices.

3.x.10 Construct nodal basis functions for $K =$ the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$, $\mathcal{P} = \mathcal{P}_2$, and $\mathcal{N} =$ evaluation at the vertices and at the midpoints of the edges.

3.x.11 Prove that the set of nodal variables

$$\Sigma_n = \{P(a), P'(a), P^{(3)}(a), \ldots, P^{(2n-1)}(a) : a = 0, 1\}$$

determine unique polynomials (in one variable) of degree $2n+1$. (For $n = 1$, this is just Hermite interpolation, as in exercise 3.x.1.)

3.x.12 Show that the nodal variables for the Argyris element described in Example 3.2.11 determine $\mathcal{P}_7$. Give a general description of the Argyris element for arbitrary degree $k \geq 5$.

3.x.13 Show that if $\mathcal{P} = \mathcal{Q}_1$, then the nodal variables depicted in Fig. 3.25 do *not* determine $\mathcal{P}$.

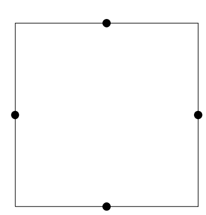

Fig. 3.25. a non-element

3.x.14 **Nonconforming piecewise linear element** Show that the edge midpoints in a triangulation can be used to parametrize the space of piecewise linear functions (in general discontinuous) that are continuous at each edge midpoint. Can you generalize this to quadratics (i.e., find a nodal basis for piecewise quadratics that are continuous at two points on each edge)?

3.x.15 **Rotated nonconforming bilinear element** Let K be the square $[-1,1] \times [-1,1]$, $\mathcal{P}$ be the space of shape functions spanned by 1, x, y and $x^2 - y^2$, and $\mathcal{N}$ consist of the evaluations of the shape functions at the four midpoints (cf. Fig. 3.25). Show that $(K, \mathcal{P}, \mathcal{N})$ is a finite element. Let $a_j(v) = (1/|e_j|) \int_{e_j} v\, ds$ be the mean value of the function v on the edge e_j of K, and $\mathcal{N}_* = \{a_1, \ldots, a_4\}$. Show that $(K, \mathcal{P}, \mathcal{N}_*)$ is also a finite element. Are these two elements interpolant equivalent?

3.x.16 **Bicubic Hermite (Bogner-Fox-Schmit) element** Prove that a tensor product cubic in two variables is uniquely determined by

$$\Sigma = \{P(a_i), \frac{\partial P}{\partial x_1}(a_i), \frac{\partial P}{\partial x_2}(a_i), \frac{\partial^2 P}{\partial x_1 \partial x_2}(a_i) : i = 1, \ldots, 4\}$$

where a_i are the rectangle vertices. Will this generate a C^1 piecewise cubic on a rectangular subdivision?

3.x.17 Let $\mathcal{I}$ be the interpolation operator associated with continuous, piecewise linears on triangles, i.e., $\mathcal{I}u = u$ at vertices. Prove that $\|\mathcal{I}\|_{C^0 \to C^0} = 1$, i.e., for any continuous function u, $\|\mathcal{I}u\|_{L^\infty} \leq \|u\|_{L^\infty}$. (Hint: where does the maximum of $|\mathcal{I}u|$ occur on a triangle?) Is this true for piecewise quadratics?

3.x.18 Let "$\angle$" denote the second derivative that is the concatenation of the directional derivatives in the two directions indicated by the line segments. Show that $\mathcal{P}_4$ is determined by (i) the value, gradient and "$\angle$" second derivative at each vertex (the directions used for "$\angle$" at each vertex are given by the edges meeting there, as shown in Fig. 3.26) and (ii) the value at each edge midpoint.

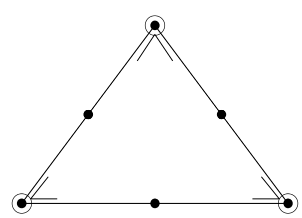

Fig. 3.26. a quartic finite element

3.x.19 Suppose that the nodes for the Lagrange element are chosen at the barycentric lattice points introduced in the proof of Proposition

3.4.5. Show that the corresponding nodal basis functions for $\mathcal{P}_k$ can be written as a product of k linear functions. (Hint: for each node determine k lines that contain all other nodes.)

3.x.20 Generalize Proposition 3.4.5 and exercise 3.x.19 to three dimensions.

3.x.21 Generalize 3.4.5 and exercise 3.x.19 to n dimensions.

3.x.22 Prove that no analog of the serendipity element exists for biquartic polynomials. (Hint: show that one can not remove all interior points.)

3.x.23 Show that the decomposition (3.6.2) is unique.

3.x.24 Develop three-dimensional Hermite and Argyris elements. Are the latter C^1?

3.x.25 Develop four-dimensional Lagrange elements.

3.x.26 Prove the "if" part of Proposition 3.4.7.

3.x.27 Show that the Hermite element is not C^1.

3.x.28 Can the derivative nodes for the Hermite elements be chosen to give an affine-equivalent family for arbitrary triangles?

3.x.29 Use (3.4.7) to prove (3.4.8).

3.x.30 Let $\mathcal{P}_k^n$ denote the space of polynomials of degree k in n variables. Prove that $\dim \mathcal{P}_k^n = \binom{n+k}{k}$, where the latter is the binomial coefficient. (Hint: show that (3.6.2) holds in n-dimensions and use this to prove that the numbers $\dim \mathcal{P}_k^n$ form Pascal's triangle.)

3.x.31 Develop three-dimensional tensor-product and serendipity elements.

3.x.32 Give conditions on rectangular subdivisions that allow the tensor-product elements to be C^0.

3.x.33 Give conditions on rectangular subdivisions that allow the bicubic Hermite elements to be C^1 (see exercise 3.x.16). Are the conditions the same as in exercise 3.x.32?

3.x.34 What conditions on simplicial subdivisions allow three-dimensional Lagrange elements to be C^0?

3.x.35 Let T be a triangle with vertices p_k $(1 \leq k \leq 3)$ and $\lambda_j \in \mathcal{P}_1$ satisfy $\lambda_j(p_k) = \delta_{jk}$ for $1 \leq j \leq 3$. Show that

$$\frac{1}{2|T|}\int_T \lambda_1^\ell \lambda_2^m \lambda_3^n \, dx = \frac{(\ell!)(m!)(n!)}{(\ell+m+n+2)!}$$

where ℓ, m and n are nonnegative integers. What is the corresponding formula for a tetrahedron?