Nonlinear Finite Elements for Continua and Structures

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element, the analyst must be aware of stability and locking characteristics of various elements. A judicious selection of an element involves factors such as the stability of the element for the problem at hand, the expected smoothness of the solution and the magnitude of deformations expected. In addition, the analyst must be aware of the complexity of nonlinear solutions. The possibility of both physical and numerical instabilities must be kept in mind and checked in a solution.

Thus the informed use of nonlinear software in both industry and research requires considerable understanding of nonlinear finite element methods. It is the objective of this book to provide this understanding and to make the reader aware of the many interesting challenges and opportunities in nonlinear finite element analysis.

1.2 RELATED BOOKS AND A BRIEF HISTORY OF NONLINEAR FINITE ELEMENTS

Several excellent texts and monographs devoted either entirely or partially to nonlinear finite element analysis have already been published. Books dealing only with nonlinear finite element analysis include Oden (1972), Crisfield (1991), Kleiber (1989), and Zhong (1993). Oden's work is particularly noteworthy since it pioneered the field of nonlinear finite element analysis of solids and structures. Recent books are Simo and Hughes (1998) and Bonet and Wood (1997). Some of the books which are partially devoted to nonlinear analysis are Belytschko and Hughes (1983), Zienkiewicz and Taylor (1991), Bathe (1996) and Cook, Malkus and Plesha (1989). These books provide useful introductions to nonlinear finite element analysis. As a companion book, a treatment of linear finite element analysis is also useful. The most comprehensive are Hughes (1987) and Zienkiewicz and Taylor (1991).

In the following, we recount a brief history of nonlinear finite element methods. This account differs somewhat from those in many other books in that it focuses more on the software than published works. In nonlinear finite element analysis, as in many endeavors in this information-computer age, the software often represents a better guide to the state-of-the-art than the literature.

Nonlinear finite element methods have many roots. Not long after the linear finite element method became known through the work of the Boeing group and the famous paper of Turner, Clough, Martin, and Topp (1956), engineers in many universities and research laboratories began extensions of the method to nonlinear, small-displacement static problems. It is difficult to convey the excitement of the early finite element community and the disdain of classical researchers for the method. For example, for many years the *Journal of Applied Mechanics* shunned papers on the finite element method because it was considered of no scientific substance. But to many, particularly engineers who had to deal with engineering problems, the promise of the finite element method was clear: it offered the possibility of dealing with the complex shapes of real designs.

The excitement in the 1960s was fueled by Ed Wilson's liberal distribution of his first programs. The first generation of these programs had no name. In many laboratories throughout the world, engineers developed new applications by modifying and extending these early codes developed at Berkeley; they had a tremendous impact on engineering and the subsequent development of finite element software. The second generation of linear programs developed at Berkeley were called SAP (Structural Analysis Program). The first nonlinear program which evolved from this work at Berkeley was NONSAP, which had capabilities for equilibrium solutions and the solution of transient problems by implicit integration.

Among the first papers on nonlinear finite element methods were Argyris (1965) and Marcal and King (1967). The number of papers soon proliferated, and software soon followed. Pedro Marcal taught at Brown University for a time, but he set up a firm to market the first nonlinear commercial finite element program in 1969; the program was called MARC and it is still a major player. At about the same time, John Swanson was developing a nonlinear finite element program at Westinghouse for nuclear applications. He left Westinghouse in 1969 to market the program ANSYS, which for many years dominated the commercial nonlinear finite element scene, although it focused more on nonlinear materials than the complete nonlinear problem.

Two other major players in the early commercial software scene were David Hibbitt and Klaus-Jürgen Bathe. Hibbitt worked with Pedro Marcal until 1972, and then co-founded HKS, which markets ABAQUS. This program has had substantial impact because it was one of the first finite element programs to introduce gateways for researchers to add elements and material models. Jürgen Bathe launched his program shortly after obtaining his PhD at Berkeley under the tutelage of Ed Wilson when he began teaching at MIT. It was an outgrowth of the NONSAP codes, and was called ADINA.

The commercial finite element programs marketed until about 1990 focused on static solutions and dynamic solutions by implicit methods. There were terrific advances in these methods in the 1970s, generated mainly by the Berkeley researchers and those with Berkeley roots: Thomas J.R. Hughes, Robert Taylor, Juan Simo, Jürgen Bathe, Carlos Felippa, Pal Bergan, Kaspar Willam, Ekerhard Ramm and Michael Ortiz are some of the prominent researchers who have been at Berkeley; it was undoubtedly the main incubator in the early years of finite elements.

Another lineage of modern nonlinear software is the explicit finite element codes. Explicit finite element methods in their early years were strongly influenced by the work in the DOE laboratories, particularly the so-called hydro-codes, Wilkins (1964).

In 1964, Costantino developed what was probably the first explicit finite element program, at the IIT Research Institute in Chicago (Costantino, 1967). It was limited to linear materials and small deformations, and computed the internal nodal forces by multiplying a banded form of the stiffness matrix by the nodal displacements. It was first run on an IBM 7040 series computer, which cost millions of dollars and had a speed of far less than 1 megaflop (million floating point operations per second) and 32 000 words of RAM. The stiffness matrix was stored on a tape and the progress of a calculation could be gauged by watching the tape drive; after every step, the tape drive would reverse to permit a read of the stiffness matrix. These and the later Control Data machines with similar specifications, the CDC 6400 and 6600, were the machines on which finite element codes were run in the 1960s. A CDC 6400 cost almost $10 million, had 32k words of memory (for storing everything including the operating system and compiler) and a real speed of about one megaflop.

In 1969, in order to sell a proposal to the Air Force, the senior author developed what has come to be known as the element-by-element technique: the computation of the nodal forces without use of a stiffness matrix. The resulting program, SAMSON, was a

two-dimensional finite element program which was used for a decade by weapons laboratories in the US. In 1972, the program was extended to fully nonlinear three-dimensional transient analysis of structures and called WRECKER. The funding was provided by a visionary program manager, Lee Ovenshire, of the US Department of Transportation, who foresaw in the early 1970s that crash testing of automobiles could be replaced by simulation.

However, it was a little ahead of its time, for at that time a simulation of a 300-element model for a 20 ms simulation took about 30 hours of computer time, which cost about $30,000, the equivalent of three years salary of an Assistant Professor. Lee Ovenshire's program funded several pioneering efforts: Hughes's work on contact-impact, Ivor McIvor's work on crush, and the research by Ted Shugar and Carly Ward on the modeling of the human head at Port Hueneme. But the Department of Transportation decided around 1975 that simulation was too expensive and all funding was redirected to testing, bringing this research effort to a screeching halt. WRECKER remained barely alive for the next decade at Ford, and the development of explicit codes by Belytschko was shifted to the nuclear safety industry at Argonne, where the code was called SADCAT and WHAMS.

Parallel work was initiated at the DOE national laboratories. In 1975, Sam Key, working at Sandia, completed HONDO, which also featured an element-by-element explicit method. The program treated both material nonlinearities and geometric nonlinearities and was carefully documented. However, this program suffered from the restrictive dissemination policies of Sandia, which did not permit codes to be released for security reasons. These programs evolved further under the work of Dennis Flanagan, a graduate of Northwestern, who named them PRONTO.

A milestone in the advancement of explicit finite element codes was John Hallquist's work at Lawrence Livermore Laboratories. John began his work in 1975, and the first release of the DYNA code was in 1976. He drew on the work which preceded his with discernment and interacted closely with many researchers from Berkeley, including Jerry Goudreau, Bob Taylor, Tom Hughes, and Juan Simo. Some of the key elements of his success were the development of contact-impact interfaces with Dave Benson, his awesome programming productivity, and the wide dissemination of the resulting codes, DYNA-2D and DYNA-3D. In contrast to Sandia, Livermore placed almost no impediments on the distribution of the program, and like Wilson's codes, John's codes were soon found in universities and government and industrial laboratories throughout the world. They were not as easy to modify, but many new ideas were developed with the DYNA codes as a testbed.

Hallquist's development of effective contact-impact algorithms (the first ones were crude compared to what is available today, but they often worked), the use of one-point quadrature elements and the high degree of vectorization made possible striking breakthroughs in engineering simulation. Vectorization has become somewhat irrelevant with the new generation of computers, but it was crucial for running large problems on the Cray machines which dominated the 1980s. The one-point quadrature elements with consistent hourglass control, to be discussed in Chapter 8, increased the speed of three-dimensional analysis by almost an order of magnitude over fully integrated three-dimensional elements.

The DYNA codes were first commercialized by a French firm, ESI, in the 1980s and called PAMCRASH, which also incorporated many routines from WHAMS. In 1989

John Hallquist left Livermore and started his own firm to distribute LSDYNA, a commercial version of DYNA.

The rapidly decreasing cost of computers and the robustness of explicit codes has revolutionized design in the past decade. The first major area of application was automotive crashworthiness, but it proliferated rapidly. In more and more industries, prototype tests are being replaced by nonlinear finite element simulations. Products such as cellphones, laptops, washing machines, chain saws, and many others are designed with the help of simulations of normal operations, drop-tests and other extreme loadings. Manufacturing processes, such as forging, sheet-metal forming, and extrusion are also simulated by finite elements. For some of these simulations, implicit methods are becoming increasingly powerful, and it is clear that both capabilities are necessary. For example, while the explicit method is probably best suited for simulating sheet metal forming operations, in the springback simulation implicit methods are more suitable.

Today, the power of implicit methods is increasing more rapidly than that of explicit methods, perhaps because they still have such a long way to go. Implicit methods for the treatment of nonlinear constraints, such as contact and friction, have been improved tremendously. Sparse iterative solvers have also become much more effective. A robust capability today requires the availability of both classes of methods.

1.3 NOTATION

Nonlinear finite element analysis represents a nexus of three fields: (1) linear finite element methods, which evolved out of matrix methods of structural analysis; (2) nonlinear continuum mechanics; and (3) mathematics, including numerical analysis, linear algebra and functional analysis (Hughes, 1996). In each of these fields a standard notation has evolved. Unfortunately, the notations are quite different, and at times contradictory or overlapping. We have tried to keep the variety of notation to a minimum and consistent within the book and with the relevant literature. To aid readers who have some familiarity with the literature on continuum mechanics or finite elements, many equations are given in matrix, tensor and indicial notation.

Three types of notation are used in this book: indicial notation, tensor notation and matrix notation. Equations relating to continuum mechanics are written in tensor and indicial notation. Equations pertaining to the finite element implementation are given in indicial or matrix notation.

1.3.1 Indicial notation In indicial notation, the components of tensors or matrices are explicitly specified. Thus a vector, which is a first-order tensor, is denoted in indicial notation by x_i, where the range of the index is the number of dimensions n_{SD}. *Indices repeated twice in a term are summed*, in conformance with the rules of Einstein notation. For example in three dimensions, if x_i is the position vector with magnitude r,

$$r^2 = x_i x_i = x_1 x_1 + x_2 x_2 + x_3 x_3 = x^2 + y^2 + z^2 \qquad (1.3.1)$$

where the second equation indicates that $x_1 = x$, $x_2 = y$, $x_3 = z$; we will usually write out the coordinates as x, y and z rather than using subscripts to avoid confusion with

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The Origins of the Finite Element Method

Je ne connais pas la source mais ça semble être une appendice d'un livre.
Tiré de la page web d'un professeur P.M. Mohite de l'Inde
Indian Institute of Techology
Kanpur

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§O.1. Introduction

This Appendix summarizes the history of structural finite elements since 1950 to date. It functions as a hub for chapter-dispersed historical references.

For exposition convenience, structural "finitelementology" may be divided into four generations that span 10 to 15 years each. There are no sharp intergenerational breaks, but noticeable change of emphasis. The following summary does not cover the conjoint evolution of Matrix Structural Analysis into the Direct Stiffness Method from 1934 through 1970. This was the subject of a separate essay [238], which is also given in Appendix H.

§O.1.1. Who Invented Finite Elements?

Not just one individual, as this historical sketch will make clear. But if the question is tweaked to: who created the FEM in everyday use? there is no question in the writer's mind: M. J. (Jon) Turner at Boeing over the period 1950–1962. He generalized and perfected the Direct Stiffness Method, and forcefully got Boeing to commit resources to it while other aerospace companies were mired in the Force Method. During 1952–53 he oversaw the development of the first continuum based finite elements. In addition to Turner, major contributors to current practice include: B. M. Irons, inventor of isoparametric models, shape functions, the patch test and frontal solvers; R. J. Melosh, who recognized the Rayleigh-Ritz link and systematized the variational derivation of stiffness elements; and E. L. Wilson, who developed the first open source (and widely imitated and distributed) FEM software.

All of these pioneers were in the aerospace industry at least during part of their careers. That is not coincidence. FEM is the confluence of three ingredients, one of which is digital computation. And only large industrial companies (as well as some government agencies) were able to afford mainframe computers during the 1950s.

Who were the popularizers? Four academicians: J. H. Argyris, R. W. Clough, H. C. Martin, and O. C. Zienkiewicz are largely responsible for the "technology transfer" from the aerospace industry to a wider range of engineering applications during the 1950s and 1960s. The first three learned the method from Turner directly or indirectly. As a consultant to Boeing in the early 1950s, Argyris, a Force Method expert then at Imperial College, received reports from Turner's group, and weaved the material into his influencial 1954 serial [22]. To Argyris goes the credit of being the first in constructing a displacement-assumed continuum element [22, p. 62].

Clough and Martin, then junior professors at U.C. Berkeley and U. Washington, respectively, spent "faculty internship" summers at Turner's group during 1952 and 1953. The result of this seminal collaboration was a celebrated paper [758], widely considered the start of the present FEM. Clough baptized the method in 1960 [136] and went on to form at Berkeley the first research group to propel the idea into Civil Engineering applications. Olek Zienkiewicz, originally an expert in finite difference methods who learned the trade from Southwell, was convinced in 1964 by Clough to try FEM. He went on to write the first textbook on the subject [821] and to organize another important Civil Engineering research group in the University of Wales at Swansea.

§O.1.2. G1: The Pioneers

The 1956 paper by Turner, Clough, Martin and Topp [758], henceforth abbreviated to TCMT, is recognized as the start of the current FEM, as used in the overwhelming majority of commercial codes. Along with Argyris' serial [22] they prototype the first generation, which spans 1950 through 1962. A panoramic picture of this period is available in two textbooks [572,596]. Przemieniecki's text is still reprinted by Dover. The survey by Gallagher [288] was influential at the time but is now difficult to access outside libraries.

The pioneers were structural engineers, schooled in classical mechanics. They followed a century of tradition in regarding structural elements as a device to transmit forces. This "element as force transducer" was the standard view in pre-computer structural analysis. It explains the use of flux assumptions to derive stiffness equations in TCMT. Element developers worked in, or interacted closely with, the aircraft industry. (As noted above, only large aerospace companies were then able to afford mainframe computers.) Accordingly they focused on thin structures built up with bars, ribs, spars, stiffeners and panels. Although the Classical Force Method dominated stress analysis during the 1950s [238], stiffness methods were kept alive by use in dynamics and vibration. It is not coincidence that Turner was a world-class expert in aeroelasticity.

§O.1.3. G2: The Golden Age

The next period spans the golden age of FEM: 1962–1972. This is the "variational generation." Melosh showed [485] that conforming displacement models are a form of Rayleigh-Ritz based on the minimum potential energy principle. This influential paper marks the confluence of three lines of research: Argyris' dual formulation of energy methods [22], the Direct Stiffness Method (DSM) of Turner [759,761], and early ideas of interelement compatibility as basis for error bounding and convergence [274,484]. G1 workers thought of finite elements as idealizations of structural components. From 1962 onward a two-step interpretation emerges: discrete elements approximate continuum models, which in turn approximate real structures.

By the early 1960s FEM begins to expand into Civil Engineering through Clough's Boeing-Berkeley connection [144,145] and had been baptized [136,138]. Reading Fraeijs de Veubeke's famous article [275] side by side with TCMT [758] one can sense the ongoing change in perspective opened up by the variational framework. The first book devoted to FEM appears in 1967 [821]. Applications to nonstructural problems had started in 1965 [820], and were treated in some depth by Martin and Carey [470].

From 1962 onwards the displacement formulation dominates. This was given a big boost by the invention of the isoparametric formulation and related tools (numerical integration, fitted natural coordinates, shape functions, patch test) by Irons and coworkers [394,397]. Low order displacement models often exhibit disappointing performance. Thus there was a frenzy to develop higher order elements. Other variational formulations, notably hybrids [573,578], mixed [353,723] and equilibrium models [275] emerged. G2 can be viewed as closed by the monograph of Strang and Fix [698], the first book to focus on the mathematical foundations.

§O.1.4. G3: Consolidation

The post-Vietnam economic doldrums are mirrored during this post-1972 period. Gone is the youthful exuberance of the golden age. This is consolidation time. Substantial effort is put into improving the stock of G2 displacement elements by tools initially labeled "variational crimes" [697], but later justified. Textbooks by Hughes [385] and Bathe [54] reflect the technology of this period. Hybrid and mixed formulations record steady progress [39]. Assumed strain formulations appear [455]. A booming activity in error estimation and mesh adaptivity is fostered by better understanding of the mathematical foundations [714].

Commercial FEM codes gradually gain importance. They provide a reality check on what works in the real world and what doesn't. By the mid-1980s there was gathering evidence that complex and high order elements were commercial flops. Exotic gadgetry interweaved amidst millions of lines of code easily breaks down in new releases. Complexity is particularly dangerous in nonlinear and dynamic analyses conducted by novice users. A trend back toward simplicity starts [457,461].

§O.1.5. G4: Back to Basics

The fourth generation begins by the early 1980s. More approaches come on the scene, notably the Free Formulation [82,86], orthogonal hourglass control [264], Assumed Natural Strain methods [57,691], stress hybrid models in natural coordinates [576,599], as well as variants and derivatives of those approaches: ANDES [225,492], EAS [671,672] and others. Although technically diverse the G4 approaches share two common objectives:

(i) Elements must fit into DSM-based programs since that includes the vast majority of production codes, commercial or otherwise.

(ii) Elements are kept simple but should provide answers of engineering accuracy with relatively coarse meshes. These were collectively labeled "high performance elements" in 1989 [219].

Two more recent trends can be noted: increased abstraction on the mathematical side,[1] and canned recipes for running commercial software on the physical side.

"Things are always at their best in the beginning," said Pascal. Indeed. By now FEM looks like an aggregate of largely disconnected methods and recipes. The blame should not be placed on the method itself, but on the community split noted in the book Preface.

§O.1.6. Precursors

As used today, FEM represents the confluence of three ingredients: Matrix Structural Analysis (MSA), variational approximation theory, and the digital computer. These came together in the early 1950. The reader should not think, however, that they simultaneouly appeared on the table through some alchemy. MSA came on the scene in the mid 1930s when desk calculators became popular, as narrated in Appendix H. And variational approximation schemes akin to those of modern FEM were proposed before digital computers. Three examples:

- The historical sketch of [470] says that "Archimedes used finite elements in determining the volume of solids." The alleged linkage is tenuous. Indeed he calculated areas, lengths and volumes of geometrical objects by dividing them into simpler ones and adding their contributions, passing to the limit as necessary. Where does "variational approximation" come in? Well, one may argue that the volume (area, length) measure of an object is a scalar functional of its geometry. Transmute "measure" into "energy" and "simpler objects" into "elements" and you capture one of the FEM tenets: the energy of the system is the sum of element energies. But for Archimedes to reach modern FEM "long is the way, and hard," since physical energy calculations require derivatives and Calculus would not be invented for 20 centuries.

[1] "If you go too far up, abstraction-wise, you run out of oxygen." (Joel Spolsky).

- In his studies leading to the creation of variational calculus, Euler divided the interval of definition of a one-dimensional functional into finite intervals and assumed a linear variation over each, defined by end values [434, p. 53]. Passing to the limit he obtained what is now called the Euler-Lagrange differential equation of variational calculus. Thus Euler deserves credit for being the first to use a piecewise linear function with discontinuous derivatives at nodes to produce, out of the hat, an ODE with second derivatives. He did not use those functions, however, to obtain an approximate value of the functional.[2]
- In the early 1940s Courant wrote an expository article [153] advocating the variational treatment of partial differential equations. The Appendix of this article contains the first FEM-style calculations on a triangular net for determining the torsional stiffness of a hollow shaft. He used piecewise linear interpolation over each triangle as Rayleigh-Ritz trial functions, and called his idea "generalized finite differences."
- A direct variational approach similar to Courant's was continued by Synge and Prager in the context of functional analysis [592] and exposed in Synge's book [712] as the "hypercircle" method.[3]
- The seminal paper by Turner et al [758] cites two immediate DSM precursors, both dated 1953, by Levy [443] and Schuerch [657]. (Only the former is available as a journal article; both have "delta wings" in the title.) From [758], p. 806: "In a recent paper Levy has presented a method of analysis for highly redundant structures that is particularly suited to the use of high-speed digital computing machines. ... The stiffness matrix for the entire structure is computed by simple summation of of the stiffness matrices of the elements of the structure."

Precursors prior to 1950 had no influence on the rapid developments of Generation 1 outlined in §O.7.2. Two crucial pieces were missing. First, and most important, was the programmable digital computer. Without computers FEM would be a curiosity, worth perhaps a footnote in an arcane book. Also missing was a *driving application* that could get the long-term attention of scientists and engineers as well as industrial resources to fund R&D work. Aerospace structural mechanics provided the driver because the necessary implementation apparatus of MSA was available since the late 1930s [282].

Matrix procedures had to be moved from desk calculators and punched-tape accounting machines to digital computers, which affluent aerospace companies were able to afford amidst Cold War paranoia. Can you imagine defense funds pouring into hypercircles or Courant's triangles? Once all pieces were in place, synergy transformed the method into a *product*, and FEM took off.

2 That would have preceded the invention of direct variational methods (Rayleigh-Ritz) for over one century, while representing also the first FEM-style calculation. A near miss indeed.

3 Curiously this book does not mention, even in passing, the use of digital computers that had already been commercially available for several years. The few numerical examples, all in 2D, are done by hand via relaxation methods.

Tenek, L.T. & Argyris J.
Finite Element Analysis for Composite Structures pg 17-25 (1998)
Volume 59 de la série Solid Mechanics & its Applications

Chapter 2

A brief history of FEM

An analysis of complex structures and other systems in a matrix formulation is now unthinkable without the finite element method. Our personal belief is that the origins of such a rich and applicable method cannot be attributed solely to one person or school of thought but rather to a synergy of various scientific developments at various research establishments. The notion of geometrical division can be traced back to the Greek natural philosopher Archimedes who in order to compute the area of a complex shape divided it into triangles and quadrilaterals whose area could be easily computed; the *assembly* of the individual areas provided the total area of the complex shape. More recently, Courant used variational and minimization arguments for the solution of physical problems. Courant [5], and Prager and Synge [6] had both proposed the concept of regional discretization which is essentially equivalent to the assumption of constant strain fields within the elements. The adaptation, however, and development of these concepts for structural analysis and other physical and technical problems was not conceptually achieved until during and shortly after World War II.

2.1 The matrix displacement method

During World War II the demand for more efficient aeronautical structures and methods for the analysis of complex structural systems provided the second author, John Argyris, with the incentive for developing the *matrix displacement method*, a concise matrix representation of the equilibrium equations governing a skeletal structure. It was wartime that necessitated this sudden explosion of knowledge. In fact in 1944, towards the end of

L. T. Tenek et al., *Finite Element Analysis for Composite Structures*

World War II, with the advent of jet propulsion, there suddenly arose the necessity of developing high subsonic speed fighters and fighter bombers. At the same time there appeared the first electromechanical calculating devices; the advent of such machines in the United States, Great Britain and Germany was a revelation to the engineers who were active in aircraft engineering and were faced with the necessity of analyzing and designing swept-back wings for the first modern high-speed fighters. In those days, practically all aircraft calculations were performed by some kind of force method in which forces or stresses appeared as the major unknowns. When the swept-back planes were to be designed it became apparent that in a structure which had practically no two parts at right angles, the force method was most inconvenient and unpractical.

In those days, John Argyris had heard of the Meteor fighter designed by Gloucester Aircraft as the answer to the German ME-262. The ME-262 had good aerodynamic performance, but suffered many failures due to the poor structural design of its swept back wings; this was a problem which the manufacturers could not overcome at that time. In those far-away days of 1944 John Argyris was working with HL Cox at the National Physical Laboratory at Teddington, London. HL Cox was then the prime figure in aircraft technology in Great Britain. In spite of the strict security precautions, they spoke about the Meteor wings and the difficulty of analyzing them. So, during the course of three brainstorming days and nights, Argyris realized that the force method was not suitable for this problem due to the great difficulty in developing the self-equilibrating systems. Consequently, he toyed with the idea of the *displacement method* and it suddenly occurred to him that the triangle was incredibly well suited to these odd swept-back wing structures –by using a number of triangles it was possible to discretize such high-speed subsonic structures! As the next step, an elementary matrix code was designed. The first structure studied using this new method was a simple wing model described by a set of 64 linear equations. It was analyzed on the electromechanical calculator at the National Physical Laboratory similar to a computing device being developed at Harvard University in the United States. Needless to say, no software was available to solve for the unknowns, so that simple, elementary solution techniques were conceived. In 2-3 days of frenzied work, displacements, strains and stresses for a loading case were obtained. HL Cox, who was aware of the on-going work, advised a direct comparison with experimental tests for a simple wing model. All were surprised when it was discovered that the maximum stresses did not deviate more than 8 to 9% from the experimental ones! This astonishing revelation percolated to the offices of the aircraft industry, and immediately a security clamp down was imposed.

By 1945 the breakdown of the continuum into triangular elements had been accomplished, and engineers had started to apply the matrix displacement method to the analysis of swept-back wings. It was not immediately realized that these developments had led to the birth of the finite element method, but the importance of the matrix displacement method was understood and the notion of ideal design and analysis of such complex structures by triangular components or elements was being grasped.

During the course of that work, it was realized that normal and shear stresses were inappropriate for analysis involving triangles; it was simpler to use three normal stresses in directions parallel to the sides of the triangle. This came to engineers as no surprise since they were working daily with strain rosettes that measured normal (direct) strains in three directions. The knowledge of the data and simple transformation rules sufficed to yield the classical cartesian strains and stresses.

After the War, in 1953, permission was granted to publish the basic idea of matrix methods, but without the major theoretical aspects of the triangular element. Aspects of the triangular element were put in two internal reports of the Department of Aeronautics of Imperial College, London, by Sydney Kelsey and S. Bagat in 1953 [7], [8]. In 1954-55, also with this restriction in mind, a series of articles was published in Aircraft Engineering by Argyris and Kelsey [9], [10] presenting in a concise matrix notation, unknown in those days, the interesting aspects of the unit load and unit displacement methods, including thermal and other initial strains. Sydney Kelsey also designed a wing-like structure with honeycomb panels with no right angle. Tests performed on this structure in the experimental laboratory of Imperial College revealed an astonishing accuracy of the stress pattern under a number of loads. This secured the acceptance of the *matrix displacement method.*

2.2 The finite element method

Essential contributions to the finite element method were made at Boeing during the summers of 1952 and 1953 under the direction of M.J. Turner [11] and led to a publication of an original paper [12]. As stated by R. W. Clough in [11]: "Mr. Turner saw the need for an improved way of taking account of the contributions of the wing skin to the stiffness of airplane wings or arbitrary configurations, and he recognized that a Ritztype procedure could be used to evaluate the contributions of individual skin elements if the wing were represented as an assemblage of such discrete structural components". It was the work conducted at Boeing that provided R.W. Clough with the inspiration for naming the method as the *finite element method.* That name appeared first in a paper presented at the 1960 ASCE Conference [13]. In the 1960s, other researchers helped to extend, disseminate and apply the finite element method.

There are other notable pioneering names: Olieg C. Zienkiewicz, Samuel Levy, Harold C. Martin, Borje Langefors, Paul H. Denke, Baudoin Fraejis De Veubeke, L. Brandeis Wehle Jr., Theodore H. Pian, Warner Lansing, Bertran Klein, John S. Archer, Robert J. Melosh, John S. Przemieniecki, Ian C. Taig, Richard H. Gallagher, Bruce Irons, etc. –the list does not claim to be exhaustive.

2.3 The natural mode finite element method

In 1962 a new approach in the context of the matrix displacement method was initiated by Argyris and Scharpf [14], [15]. Experimentation with problems involving large displacements, strains and inelastic behavior led to the basic concepts of geometrical stiffness and the idea of *natural modes.* These natural modes essentially represented invariant fields, including both rigid body and pure straining measures, that were used to describe the elemental kinematics. Since then this fundamental idea has evolved significantly, and has facilitated the computer simulation of large and complex structures in the linear and nonlinear regimes. In addition, it has provided an insight into the fundamental behavior of structures undergoing small and large displacements. It is now possible to create finite elements based on *rigid body and straining modes of deformation.* The technique is particularly suited to the creation of simple truss, beam, plate and shell triangular elements, and tetrahedron volume elements for large scale and fast engineering computations.

Although, over the years, many finite elements have been developed on these principles, the natural mode method has only recently been applied by Tenek and Argyris [16], [17] to the analysis of laminated beam and shell composite structures through a general formulation able to treat, as special cases, isotropic and sandwich plates and shells. The culmination of four years of research conducted on this field is presented in this text in a simple and comprehensible manner for linear and nonlinear statics.

2.4 The basic ideas of FEM

In the article "A brief history of the beginning of the finite element method" Gupta and Meek [18] presented summaries of the works of several authors associated with the invention of the finite element method and its application to structural mechanics. They discerned five groups of papers which may be considered as the starting inspirations of FEM. They are the papers by Courant [5], Argyris and Kelsey [9], [10], Turner, Clough, Martin and Top [12], Clough [13], and Zienkiewicz and Cheung [19].

Courant

Courant [5] developed the idea of the minimization of a functional using linear approximation over subregions. He specified the values at discrete points similar to the node points of a finite element mesh. In his paper he shows a mesh subdivision up to 9 approximate points to solve the St Venant's torsion of a square hollow box. He considered the potential energy U of the system and used the condition of stationary potential energy which he expressed as

$$\delta(V + U) = 0. \tag{2.1}$$

He specified the shear stresses in the shaft as first derivatives of a stress function ϕ, and stated that if ϕ is described in terms of a number of discrete parameters a_i, the stationary condition (2.1) leads to a set of linear equations

$$\frac{\partial(V + U)}{\partial a_i} = 0. \tag{2.2}$$

Courant also applied the Rayleigh-Ritz method to create a functional for ϕ with two unknowns.

Courant did not fully clarify his piecewise linear approximation to the ϕ surface nor did he give any mathematical details of this approximation. He clearly indicated, however, a procedure which could be used for the minimization of the total potential energy for the torsion problem.

Argyris and Kelsey

In the series of papers "Energy Theorems and Structural Analysis" Argyris and Kelsey [9], [10] developed the matrix theory of structures for discrete elements. They developed the concepts of flexibility and stiffness and provided equations which have become standard in structural mechanics. They include

$$\begin{aligned} \text{flexibility}: \quad [F] &= \int_V [b]^{\mathrm{T}}[f][b]dV, \\ \text{stiffness}: \quad [K] &= \int_V [a]^{\mathrm{T}}[k][a]dV, \end{aligned} \tag{2.3}$$

in which $[f], [k]$ are constitutive relations and $[b], [a]$ are stress-force and strain-displacement relationships, respectively. They applied their theory to a rectangular panel for the case of nodal displacements which vary linearly along the edges of the element and calculated its stiffness matrix; the first element in plane stress using interpolation functions in terms of nodal displacements was developed! The panel had an 8×8 stiffness matrix which was expressed as a sum of shear and direct strain matrices:

$$[K] = [K_s] + [K_d]. \tag{2.4}$$

Thus Argyris and Kelsey developed the rectangular panel stiffness matrix in plane stress using element interpolation functions in terms of nodal displacements. They also showed that, in the context of aircraft skin models, the triangular panel behavior may well be approximated by using energy minimization procedures which were shown later to provide a firm basis for finite element formulation.

Turner, Clough, Martin and Top

The pioneering paper of Turner *et al.* [12] discusses the truss member and derives its stiffness matrix in global coordinates in the form:

$$[K] = \frac{AE}{L} \begin{bmatrix} \lambda^2 & -\lambda^2 & \lambda\mu & -\lambda\mu \\ -\lambda^2 & \lambda^2 & -\lambda\mu & \lambda\mu \\ \lambda\mu & -\lambda\mu & \mu^2 & -\mu^2 \\ -\lambda\mu & \lambda\mu & -\mu^2 & \mu^2 \end{bmatrix}, \tag{2.5}$$

where λ, μ are member direction cosines, L is the member length and A its area of cross-section. The paper continues with a discussion on rectangular plate elements and then turns to triangular elements, stating that the latter will be used as the basic building block for calculating stiffness matrices for plates of arbitrary shape. It then proceeds to the study of the triangular element in plane stress. It starts by assuming constant strains which are integrated to yield the displacements u, v. Then, it expresses the relation between stresses σ_{xx}, σ_{yy} and σ_{xy} and the nodal displacements in the form

$$\sigma = [S]\{\delta\}. \tag{2.6}$$

The nodal forces were obtained from the three stresses via

$$F = [T]\{\sigma\} = [T][S]\{\delta\}. \tag{2.7}$$

The element stiffness matrix was defined as

$$[K] = [T][S]. \tag{2.8}$$

Note that equation (2.8) is equivalent to the second of equations (2.3) developed by Argyris and Kelsey. The Turner paper also addresses the question of convergence.

Clough

The name finite element method is attributed to Clough [13], [11], who was a coauthor of the original paper by Turner [12]. In [11], [13], he outlined how he first invented the name finite element method because he wished to show the distinction between the continuum analysis and the matrix method of structural analysis. In [20] he outlined the research program undertaken at Boeing Company in 1952-1953 for the calculation of the flexibility coefficients for low aspect ratio wing structures for dynamic analysis. He extended Turner's work from 1957 onwards; continued convergence studies on stress components; and popularized the ideas of the finite element method.

Zienkiewicz and Cheung

The development of the non-structural applications by means of minimization of the total potential energy of a system is developed systematically for the first time in the paper by Zienkiewicz and Cheung [21], in which heat transfer and St Venant's torsion of prismatic shafts are analyzed. In this paper they set up the approximation to the functional in terms of the nodal values of the triangular domain, into which the region is subdivided.

The function minimization techniques, originally discussed by Courant, were clarified by Zienkiewicz in 1965; this opened the way to the analysis of field problems by the FEM. A first book on the FEM was published in 1957 [19] with new editions appearing in recent years [22].

Zienkiewicz expressed his personal views on the origins, milestones and directions of the finite element method in [23]. He stated that the finite element method was made possible only by the advent of the electronic digital computer, and discussed the "variational" approaches via extremum principles.

Thus if the strain field in an elastic continuum is defined by a suitable operator $\boldsymbol{S}$ acting on the displacements $\boldsymbol{u}$ as

$$\boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{u}, \tag{2.9}$$

with the corresponding stresses given as

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}, \tag{2.10}$$

where $\boldsymbol{D}$ is a matrix of elastic constants, then the finite element solution sought could be obtained by the minimization of the potential energy

$$\boldsymbol{\Pi} = \frac{1}{2}\int_{\Omega} \boldsymbol{\varepsilon}^T \boldsymbol{D}\boldsymbol{\varepsilon} d\Omega - \int_{\Gamma_t} \bar{\boldsymbol{t}}^T \boldsymbol{u}\, d\Gamma - \int_{\Omega} \boldsymbol{b}^T \boldsymbol{u}\, d\Omega. \tag{2.11}$$

The displacement field is approximated as

$$\boldsymbol{u}^h = \boldsymbol{N}\bar{\boldsymbol{u}}, \tag{2.12}$$

where $\bar{\boldsymbol{u}}$ are nodal values of $\boldsymbol{u}$ or other parameters satisfying prescribed displacements on the boundary $\boldsymbol{\Gamma}_t$; $\bar{\boldsymbol{t}}$ are the prescribed tractions on the boundary $\boldsymbol{\Gamma}_t$; and $\boldsymbol{b}$ are the body forces. The functions $\boldsymbol{N}$ are given in terms of the coordinates and are known as *shape* or *basis* functions. The minimization of (2.11) leads to a set of *discrete* algebraic equations of the form

$$\boldsymbol{K}\bar{\boldsymbol{u}} = \boldsymbol{f}, \tag{2.13}$$

where

$$K = \sum K_e, \quad K_e = \int_{\Omega_e} (SN)^T D(SN) d\Omega, \tag{2.14}$$

and

$$f = \sum \mathrm{f}_e, \quad f_e = \int_{\Omega_e} N^T b \, d\Omega + \int_{\Gamma_t^e} N^T \bar{t} \, d\Gamma. \tag{2.15}$$

The domains Ω_e and Γ^e correspond to *elements* into which the whole continuum problem is divided. Equations (2.14), (2.15) are used to generate element stiffness coefficients and forces providing that the approximation shape functions of Eq. (2.12) are defined on a local basis. These equations summarize the basic philosophy of *classical finite element methods.* Zienkiewicz points out that such a derivation of the finite element procedure is a particular case of the approach introduced much earlier by Rayleigh [24] and Ritz [25]. The main difference lies in the use of the local shape functions $\boldsymbol{N}$ which yield a banded structure of the assembled stiffness matrix $\boldsymbol{K}$ and preserve the local assembly structure of the matrix equations.